文章编号:1007－6735(2019)03－0214－10 DOI: 10.13255/j.cnki.jusst.2019.03.002

分数阶微分方程组边值问题解的存在性与唯一性

郭莉莉， 刘锡平， 贾 梅， 蹇星月
（上海理工大学 理学院，上海 200093）

摘要：研究了一类高阶 Riemann-Liouville 分数阶微分方程组边值问题。通过 Laplace 变换的方法得到边值问题解的积分表达形式，建立了边值问题解的存在性定理和存在唯一性定理，利用 Leray-Schauder 抉择证明了解的存在性定理，运用 Banach 压缩映射原理证明了解的存在唯一性定理。最后给出 2 个例子说明所得结论的适用性。

关键词：微分方程组；边值问题；Riemann-Liouville 分数阶导数；Laplace 变换；不动点定理
中图分类号：O 175.8 **文献标志码：**A

Existence and Uniqueness of Solutions of Boundary Value Problems for Fractional Differential Equation Systems

GUO Lili， LIU Xiping， JIA Mei， JIAN Xingyue
(*College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China*)

Abstract: A class of boundary value problems for high order Riemann-Liouville fractional differential equation systems was studied. By the Laplace transform method, the integral expression of the boundary value problems was obtained. The existence theorem and the existence and uniqueness theorem for the solutions of the boundary value problems were established and proved by using the Leray-Schauder alternative and the Banach contraction mapping principle, respectively. Two examples were given to illustrate the main results.

Keywords: *differential equation system*; *boundary value problem*; *Riemann-Liouville fractional derivative*; *Laplace transform*; *fixed point theorem*

1 问题的提出

在工程技术和科学研究中，有许多现象是由微分方程来描述的。随着科学技术的发展，人们对分数阶微分方程的研究越来越多，取得了很多成果[1-13]。

微分方程组通常用来描述涉及到多个状态变量的运动系统，文献[14-15]研究了具有 Caputo 导

收稿日期: 2018−06−26
基金项目: 国家自然科学基金资助项目（11171220）
第一作者: 郭莉莉(1993−)，女，硕士研究生．研究方向：常微分方程理论与应用．E-mail：1259094845@qq.com
通信作者: 刘锡平(1962−)，男，教授．研究方向：常微分方程理论与应用．E-mail：xipingliu@usst.edu.cn

数的分数阶微分方程组边值问题解的存在性。

本文研究一类高阶 Riemann-Liouville 分数阶微分方程组边值问题。

$$\begin{cases}\mathrm{D}^{\alpha}\boldsymbol{U}(t)+\boldsymbol{\Lambda}\mathrm{D}^{\alpha-1}\boldsymbol{U}(t)=\boldsymbol{F}(t,\boldsymbol{U}(t)),\ 0<t<1\\ u_1(\eta_1)=u_2(\eta_2)=u_3(\eta_3)=\cdots=u_n(\eta_n)=0\\ u_1(1)=u_2(1)=u_3(1)=\cdots=u_n(1)=0\end{cases}\qquad(1)$$

其中，

$$\boldsymbol{U}(t)=\begin{pmatrix}u_1(t)\\u_2(t)\\\vdots\\u_n(t)\end{pmatrix},\ \mathrm{D}^{\alpha}\boldsymbol{U}(t)=\begin{pmatrix}\mathrm{D}^{\alpha}u_1(t)\\\mathrm{D}^{\alpha}u_2(t)\\\vdots\\\mathrm{D}^{\alpha}u_n(t)\end{pmatrix}$$

$$\boldsymbol{F}=\begin{pmatrix}f_1\\f_2\\\vdots\\f_n\end{pmatrix},\ \boldsymbol{\Lambda}=\begin{pmatrix}\lambda_1&0&0&\cdots&0\\0&\lambda_2&0&\cdots&0\\0&0&\lambda_3&\cdots&0\\\vdots&\vdots&\vdots&&\vdots\\0&0&0&\cdots&\lambda_n\end{pmatrix}$$

D^{α}表示 Riemann-Liouville 导数，$1<\alpha<2, 0<\lambda_i<1, 0<\eta_i<1, f_i\in C(I\times\mathbb{R}^n,\mathbb{R})$，$t\in[0,1], i=1,2,\cdots,n$。

2　预备知识与引理

为了后面证明的需要，现给出一些定义和引理。

定义 1[16]　设函数$f(t)$当$t\geqslant 0$时有定义，且积分$\int_0^{+\infty}f(t)\mathrm{e}^{-st}\,\mathrm{d}t$（$s$是一个复变量）在 s 的某一个域内收敛，则由此积分所确定的函数$F(s)=\int_0^{+\infty}f(t)\mathrm{e}^{-st}\mathrm{d}t$称为函数$f(t)$的 Laplace 变换，记$F(s)=\mathcal{L}(f(t))$，$F(s)$称为$f(t)$的像函数. 在相同条件下，称

$$f(t)=\mathcal{L}^{-1}(F(s))=\frac{1}{2\pi\mathrm{i}}\int_{s-\mathrm{i}\infty}^{s+\mathrm{i}\infty}F(p)\mathrm{e}^{pt}\mathrm{d}p$$

为$F(s)$的 Laplace 逆变换。

引理 1[16]　若$\mathcal{L}(f_1(t))=F_1(s),\mathcal{L}(f_2(t))=F_2(s)$,则

$$\mathcal{L}(f_1(t)*f_2(t))=F_1(s)F_2(s)$$
$$\mathcal{L}^{-1}(F_1(s)*F_2(s))=f_1(t)f_2(t)$$

定义 2[16]　函数 $u:\mathbb{R}_+\to\mathbb{R}$的 α阶 Riemann-Liouville 分数阶积分定义为

$$\mathrm{I}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}u(\tau)\mathrm{d}\tau$$

对任意的$\alpha>0$，右端积分在$\mathbb{R}_+$上逐点可积，$\Gamma(\cdot)$是 Gamma 函数。

定义 3[17]　函数 $u:\mathbb{R}_+\to\mathbb{R}$的 α阶 Riemann-Liouville 分数阶导数定义为

$$\mathrm{D}^{\alpha}u(t)=\mathrm{D}^n\mathrm{I}^{n-\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t^n}\right)^n\int_0^t\frac{u(\tau)}{(t-\tau)^{\alpha+1-n}}\mathrm{d}\tau$$

对任意的$\alpha>0$，右端积分在$\mathbb{R}_+$上逐点可积，n 为大于或等于α的最小整数。

引理 2[17]　对任意的$\alpha>0$, 函数$u:\mathbb{R}_+\to\mathbb{R}$的$\alpha$阶 Laplace 变换公式为

$$\mathcal{L}(\mathrm{D}^{\alpha}u)(s)=s^{\alpha}\mathcal{L}(u)(s)-\sum_{j=1}^{l}d_js^{j-1},\quad l-1\leqslant\alpha\leqslant l, l\in\mathbb{N}$$

$$d_j=(\mathrm{D}^{\alpha-j}u)(0^+),\quad j=1,2,\cdots,l$$

定义 4[17]　令$\alpha,\beta>0$，函数$E_{\alpha,\beta}$的定义为

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$$

当级数收敛时，称级数是关于参数α,β的二元 Mittag-Leffler 函数。

引理 3[17]　Mittag-Leffler 函数的 Laplace 变换公式为

$$\mathcal{L}\left(t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm at^{\alpha})\right)=\int_0^{\infty}\mathrm{e}^{-st}t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm at^{\alpha})\mathrm{d}t=\frac{k!s^{\alpha-\beta}}{(s^{\alpha}ma)^{k+1}},\quad \mathrm{Re}(s)>|a|^{\frac{1}{\alpha}}$$

引理 4[17]　广义 Mittag-Leffler 导数定义为

$$E_{\alpha,\beta}^{(n)}(z)=\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n\left(E_{\alpha,\beta}^{n+1}(z)\right)=n!E_{\alpha,\beta+\alpha n}^{n+1}(z),$$
$$z\in\mathbb{C},\ \alpha,\beta,\rho\in\mathbb{C},\ R(\alpha)>0$$

其中，

$$E_{\alpha,\beta}^{\rho}(z)=\sum_{k=0}^{\infty}\frac{(\rho)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}$$
$$(\rho)_k=\rho(\rho+1)\cdots(\rho+k-1)$$

引理 5　设$0<\lambda_1<1$, 则级数

$$\sum_{n=0}^{\infty}\frac{(1-\lambda_1)^n}{n!}t^{n+\alpha-1}E_{1,\alpha}^{(n)}(-t)=\sum_{n=0}^{\infty}\frac{(1-\lambda_1)^n}{n!}t^{n+\alpha-1}\sum_{k=0}^{\infty}\frac{\Gamma(n+k+1)}{\Gamma(n+k+\alpha)}\frac{(-t)^k}{k!}\triangleq g_{11}(t)$$

$$\sum_{n=0}^{\infty}\frac{(1+\lambda_1)(1-\lambda_1)^n}{n!}t^{n+\alpha-1}E_{1,\alpha}^{(n)}(-t)=\sum_{n=0}^{\infty}\frac{(1+\lambda_1)(1-\lambda_1)^n}{n!}t^{n+\alpha-1}\sum_{k=0}^{\infty}\frac{\Gamma(n+k+1)}{\Gamma(n+k+\alpha)}\frac{(-t)^k}{k!}\triangleq g_{12}(t)$$

$$t^{2-\alpha}\sum_{n=0}^{\infty}\frac{(1-\lambda_1)^n}{n!}t^{n+\alpha-2}E_{1,\alpha-1}^{(n)}(-t)=\sum_{n=0}^{\infty}\frac{(1-\lambda_1)^n}{n!}t^n\sum_{k=0}^{\infty}\frac{\Gamma(n+k+1)}{\Gamma(n+k+\alpha-1)}\frac{(-t)^k}{k!}\triangleq t^{2-\alpha}g_{13}(t)$$

在$[0,1]$上一致收敛。证明从略。

因为，

$$\lim_{t\to0^+} t^{2-\alpha}g_{13}(t)=$$

$$\lim_{t\to0^+}\left(\sum_{n=0}^{\infty}\frac{(1-\lambda_1)^n}{n!}t^n\sum_{k=0}^{\infty}\frac{\Gamma(n+k+1)}{\Gamma(n+k+\alpha-1)}\frac{(-t)^k}{k!}\right)=$$

$$\frac{1}{\Gamma(\alpha-1)}$$

所以，$t^{2-\alpha}g_{13}(t)|_{t=0}=\dfrac{1}{\Gamma(\alpha-1)}$，即函数 $t^{2-\alpha}g_{13}(t)$ 在 $t=0$ 时连续。

引理 6 函数 $g_{11}(t), g_{12}(t), g_{13}(t)$ 的性质：

a. $0<g_{11}(t)<\dfrac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_1)}$;

b. 函数 $g_{11}(t), g_{12}(t), t^{2-\alpha}g_{13}(t)$ 在 $[0, 1]$ 上连续。

证明 **a**. 由引理 5 可知，$g_{11}(t)>0$ 显然成立。

$$g_{11}(t)=\sum_{n=0}^{\infty}\frac{(1-\lambda_1)^n}{n!}t^{n+\alpha-1}\sum_{k=0}^{\infty}\frac{\Gamma(n+k+1)}{\Gamma(n+k+\alpha)}\frac{(-t)^k}{k!}=$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{t^{k+\alpha-1}}{\Gamma(k+\alpha)}+$$

$$\sum_{n=1}^{\infty}\frac{(1-\lambda_1)^n}{n!}t^{n+\alpha-1}\sum_{k=0}^{\infty}\frac{\Gamma(n+k+1)}{\Gamma(n+k+\alpha)}\frac{(-t)^k}{k!}$$

由引理 5 可知，级数 $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{t^{k+\alpha-1}}{\Gamma(k+\alpha)}$ 关于 t 在 $[0,1]$ 上一致收敛，

因为，

$$\frac{t^{k+\alpha}}{\Gamma(k+\alpha+1)}\frac{\Gamma(k+\alpha)}{t^{k+\alpha-1}}=\frac{t}{k+\alpha}<1$$

所以，函数列 $\left\{\dfrac{t^{k+\alpha-1}}{\Gamma(k+\alpha)}\right\}$ 关于 k 单调递减。

结合 Leibniz 判别法可知，

$$0<\sum_{k=0}^{\infty}(-1)^k\frac{t^{k+\alpha-1}}{\Gamma(k+\alpha)}<\frac{t^{\alpha-1}}{\Gamma(\alpha)}<\frac{1}{\Gamma(\alpha)}$$

由引理 5 可知，级数

$$\sum_{n=1}^{\infty}\frac{(1-\lambda_1)}{n!}t^{n+\alpha-1}\sum_{k=0}^{\infty}\frac{\Gamma(n+k+1)}{\Gamma(n+k+\alpha)}\frac{(-t)^k}{k!}<$$

$$\sum_{n=0}^{\infty}\frac{(t(1-\lambda_1))^n}{n!}=\mathrm{e}^{t(1-\lambda_1)}<\mathrm{e}^{(1-\lambda_1)}$$

即证 $0<g_{11}(t)<\dfrac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_1)}$。

b. 函数 $g_{11}(t), g_{12}(t), t^{2-\alpha}g_{13}(t)$ 在 $[0, 1]$ 上连续，显然成立。

证毕。

引理 7 设 $1<\alpha<2, \varDelta_1=g_{12}(\eta_1)g_{13}(1)-g_{12}(1)g_{13}(\eta_1)\neq 0$，则对任意 $h_1\in C(I,\mathbb{R})$，边值问题

$$\begin{cases}\mathrm{D}^{\alpha}u_1(t)+\lambda_1\mathrm{D}^{\alpha-1}u_1(t)=h_1(t), & 0<t<1\\ u_1(\eta_1)=0, \quad u_1(1)=0\end{cases}\tag{2}$$

存在唯一解

$$u_1(t)=\int_0^t g_{11}(t-\tau)h_1(\tau)\mathrm{d}\tau+$$

$$\frac{g_{12}(1)g_{13}(t)-g_{13}(1)g_{12}(t)}{\varDelta_1}\int_0^{\eta_1}g_{11}(\eta_1-\tau)h_1(\tau)\mathrm{d}\tau+$$

$$\frac{g_{13}(\eta_1)g_{12}(t)-g_{12}(\eta_1)g_{13}(t)}{\varDelta_1}\int_0^1 g_{11}(1-\tau)h_1(\tau)\mathrm{d}\tau\tag{3}$$

证明 由引理 2 可知，

$$\mathcal{L}(\mathrm{D}^{\alpha}u_1(t))(s)=s^{\alpha}\mathcal{L}(u(t))(s)-d_{11}-d_{12}s$$

$$\mathcal{L}(\mathrm{D}^{\alpha-1}u_1(t))(s)=s^{\alpha-1}\mathcal{L}(u(t))(s)-d_{11}$$

所以，对 $\mathrm{D}^{\alpha}u_1(t)+\lambda_1\mathrm{D}^{\alpha-1}u_1(t)=h_1(t)$ 进行 Laplace 变换，可得

$$(s^{\alpha}+\lambda_1 s^{\alpha-1})\mathcal{L}(u_1(t))(s)-(1+\lambda_1)d_{11}-d_{12}s=\mathcal{L}(h_1(t))(s)$$

整理得到

$$\mathcal{L}(u_1(t))(s)=\frac{1}{s^{\alpha}+\lambda_1 s^{\alpha-1}}\mathcal{L}(h_1(t))(s)+$$

$$\frac{1+\lambda_1}{s^{\alpha}+\lambda_1 s^{\alpha-1}}d_{11}+\frac{s}{s^{\alpha}+\lambda_1 s^{\alpha-1}}d_{12}$$

又因为，

$$\frac{1}{s^{\alpha}+\lambda_1 s^{\alpha-1}}=\frac{s^{1-\alpha}}{s+\lambda_1+1-1}=\frac{s^{1-\alpha}}{1+s}\frac{1}{1+\dfrac{\lambda_1-1}{1+s}}=$$

$$\frac{s^{1-\alpha}}{1+s}\sum_{n=0}^{\infty}\frac{(1-\lambda_1)^n}{(1+s)^n}=\sum_{n=0}^{\infty}\frac{(1-\lambda_1)^n s^{1-\alpha}}{(1+s)^{n+1}}$$

$$\frac{s}{s^{\alpha}+\lambda_1 s^{\alpha-1}}=\frac{s^{2-\alpha}}{s+\lambda_1+1-1}=\frac{s^{2-\alpha}}{1+s}\frac{1}{1+\dfrac{\lambda_1-1}{1+s}}=$$

$$\frac{s^{2-\alpha}}{1+s}\sum_{n=0}^{\infty}\frac{(1-\lambda_1)^n}{(1+s)^n}=\sum_{n=0}^{\infty}\frac{(1-\lambda_1)^n s^{2-\alpha}}{(1+s)^{n+1}}$$

所以，

$$\mathcal{L}(u_1(t))(s)=\sum_{n=0}^{\infty}\frac{(1-\lambda_1)^n s^{1-\alpha}}{(1+s)^{n+1}}\mathcal{L}(h_1(t))(s)+$$

$$\sum_{n=0}^{\infty}\frac{(1+\lambda_1)(1-\lambda_1)^n s^{1-\alpha}}{(1+s)^{n+1}}d_{11}+$$

$$\sum_{n=0}^{\infty}\frac{(1-\lambda_1)^n s^{2-\alpha}}{(1+s)^{n+1}}d_{12}\tag{4}$$

由引理 3 可知，

$$\frac{s^{1-\alpha}}{(1+s)^{n+1}}=\frac{1}{n!}\mathcal{L}\left(t^{n+\alpha-1}E_{1,\alpha}^{(n)}(-t)\right)(s)$$

$$\frac{s^{2-\alpha}}{(1+s)^{n+1}}=\frac{1}{n!}\mathcal{L}\left(t^{n+\alpha-2}E_{1,\alpha-1}^{(n)}(-t)\right)(s) \tag{5}$$

将式(5)代入式(4)并整理，得到

$$\begin{aligned}\mathcal{L}(u_1(t))(s)=&\mathcal{L}\left(\sum_{n=0}^{\infty}\frac{(1-\lambda_1)^n}{n!}t^{n+\alpha-1}E_{1,\alpha}^{(n)}(-t)\right)(s)\mathcal{L}(h_1(t))(s)+\\&d_{11}\mathcal{L}\left(\sum_{n=0}^{\infty}\frac{(1+\lambda_1)(1-\lambda_1)^n}{n!}t^{n+\alpha-1}E_{1,\alpha}^{(n)}(-t)\right)(s)+\\&d_{12}\mathcal{L}\left(\sum_{n=0}^{\infty}\frac{(1-\lambda_1)^n}{n!}t^{n+\alpha-2}E_{1,\alpha-1}^{(n)}(-t)\right)(s)=\\&\mathcal{L}(g_{11}(t))(s)\mathcal{L}(h_1(t))(s)+d_{11}\mathcal{L}(g_{12}(t))(s)+\\&d_{12}\mathcal{L}(g_{13}(t))(s)=\mathcal{L}(g_{11}(t)*h_1(t))(s)+\\&d_{11}\mathcal{L}(g_{12}(t))(s)+d_{12}\mathcal{L}(g_{13}(t))(s)=\\&\mathcal{L}\left(\int_0^t g_{11}(t-\tau)h_1(\tau)\mathrm{d}\tau\right)(s)+\\&d_{11}\mathcal{L}(g_{12}(t))(s)+d_{12}\mathcal{L}(g_{13}(t))(s)\end{aligned} \tag{6}$$

对式(6)两边进行 Laplace 逆变换，得到

$$u_1(t)=\int_0^t g_{11}(t-\tau)h_1(\tau)\mathrm{d}\tau+d_{11}g_{12}(t)+d_{12}g_{13}(t) \tag{7}$$

将边界条件$u_1(1)=0, u_1(\eta_1)=0$代入式(7)，得到

$$\int_0^{\eta_1} g_{11}(\eta_1-\tau)h_1(\tau)\mathrm{d}\tau+d_{11}g_{12}(\eta_1)+d_{12}g_{13}(\eta_1)=0$$

$$\int_0^{1} g_{11}(1-\tau)h_1(\tau)\mathrm{d}\tau+d_{11}g_{12}(1)+d_{12}g_{13}(1)=0$$

设$\varDelta_1\neq 0$时，解得

$$d_{11}=\frac{1}{\varDelta_1}\left(g_{13}(\eta_1)\int_0^1 g_{11}(1-\tau)h_1(\tau)\mathrm{d}\tau-g_{13}(1)\int_0^{\eta_1} g_{11}(\eta_1-\tau)h_1(\tau)\mathrm{d}\tau\right)$$

$$d_{12}=\frac{1}{\varDelta_1}\left(g_{12}(1)\int_0^{\eta_1} g_{11}(\eta_1-\tau)h_1(\tau)\mathrm{d}\tau-g_{12}(\eta_1)\int_0^1 g_{11}(1-\tau)h_1(\tau)\mathrm{d}\tau\right)$$

将 d_1，d_2 带入式(7)，可得式(3)成立。

证毕。

由引理 7 即可得到引理 8。

引理 8　边值问题(1)等价于积分方程

$$\begin{aligned}u_i(t)=&\int_0^t g_{i1}(t-\tau)f_i(\tau,u_1(\tau),\cdots,u_n(\tau))\mathrm{d}\tau+\\&\frac{g_{i2}(1)g_{i3}(t)-g_{i3}(1)g_{i2}(t)}{\varDelta_i}\cdot\\&\int_0^{\eta_i} g_{i1}(\eta_i-\tau)f_i(\tau,u_1(\tau),\cdots,u_n(\tau))\mathrm{d}\tau+\\&\frac{g_{i3}(\eta_i)g_{i2}(t)-g_{i2}(\eta_i)g_{i3}(t)}{\varDelta_i}\cdot\\&\int_0^{1} g_{i1}(1-\tau)f_i(\tau,u_1(\tau),\cdots,u_n(\tau))\mathrm{d}\tau\end{aligned}$$

其中，

$$i=1,2,\cdots,n,\ \varDelta_i=g_{i2}(\eta_i)g_{i3}(1)-g_{i2}(1)g_{i3}(\eta_i)\neq 0$$

$$g_{i1}(t)=\sum_{n=0}^{\infty}\frac{(1-\lambda_i)^n}{n!}t^{n+\alpha-1}E_{1,\alpha}^{(n)}(-t)$$

$$g_{i2}(t)=\sum_{n=0}^{\infty}\frac{(1+\lambda_i)(1-\lambda_i)^n}{n!}t^{n+\alpha-1}E_{1,\alpha}^{(n)}(-t)$$

$$g_{i3}(t)=\sum_{n=0}^{\infty}\frac{(1-\lambda_i)^n}{n!}t^{n+\alpha-2}E_{1,\alpha-1}^{(n)}(-t)$$

引理 9[18]（Leray-Shauder 抉择）　令E是 Banach 空间，假设$T:E\to E$是全连续算子，令$V(T)=\{x\in E:$ 存在$\mu\in(0,1)$, 使得$x=\mu T(x)\}$，则集合$V(T)$是无界集或者算子 T 至少存在 1 个不动点。

3　解的存在性与唯一性

空间$E_1=C_{2-\alpha}^0[0,1]=\{u\in C(0,1]:\lim\limits_{t\to 0^+}t^{2-\alpha}u(t)$存在}，以范数$\|u\|=\sup\limits_{t\in(0,1]}t^{2-\alpha}|u(t)|$构成了 Banach 空间。所以，$E=\underbrace{E_1\times E_1\times\cdots\times E_1}_{n}$以范数$\|\boldsymbol{U}\|_E=\|u_1\|+\|u_2\|+\cdots+\|u_n\|$构成了 Banach 空间。

设$\varDelta_i\neq 0, i=1,2,\cdots,n$，为了方便，给出记号：

$$l_{i1}=\sup_{t\in(0,1]}\left|t^{2-\alpha}\frac{g_{i2}(1)g_{i3}(t)-g_{i3}(1)g_{i2}(t)}{\varDelta_i}\right|$$

$$l_{i2}=\sup_{t\in(0,1]}\left|t^{2-\alpha}\frac{g_{i3}(\eta_i)g_{i2}(t)-g_{i2}(\eta_i)g_{i3}(t)}{\varDelta_i}\right|$$

由引理 8 可知，对任意的$\boldsymbol{U}(t)=(u_1(t),u_2(t),\cdots,u_n(t))^{\mathrm{T}}\in E$, 定义算子

$$T(\boldsymbol{U})(t)=(T_1(\boldsymbol{U})(t),\cdots,T_n(\boldsymbol{U})(t))^{\mathrm{T}}$$

即

$$T_i(\boldsymbol{U}(t))=\int_0^t g_{i1}(t-\tau)f_i(\tau,\boldsymbol{U}(\tau))\,\mathrm{d}\tau+\frac{g_{i2}(1)g_{i3}(t)-g_{i3}(1)g_{i2}(t)}{\varDelta_i}\cdot\int_0^{\eta_i}g_{i1}(\eta_i-\tau)f_i(\tau,\boldsymbol{U}(\tau))\mathrm{d}\tau+\frac{g_{i3}(\eta_i)g_{i2}(t)-g_{i2}(\eta_i)g_{i3}(t)}{\varDelta_i}\cdot\int_0^1 g_{i1}(1-\tau)f_i(\tau,\boldsymbol{U}(\tau))\mathrm{d}\tau \tag{8}$$

其中，$i=1,2,\cdots,n$。所以，算子$T:E\to E$。

为了证明的方便，给出记号：

$$\gamma_i=\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)},\theta_i=1+\eta_i l_{i1}+l_{i2} \tag{9}$$

$$\varPhi_0=\gamma_1\theta_1k_{10}+\gamma_2\theta_2k_{20}+\cdots+\gamma_n\theta_nk_{n0}$$

$$\varPhi_i=\frac{1}{\alpha-1}(\gamma_1\theta_1k_{1i}+\gamma_2\theta_2k_{2i}+\cdots+\gamma_n\theta_nk_{ni}) \tag{10}$$

定理 1 设$\varDelta_i\neq0$, 存在实数$k_{ij}(i,j=1,2,\cdots,n)$且$k_{i0}>0$，使得

$$|f_i(t,u_1,\cdots,u_n)|\leqslant k_{i0}+k_{i1}|u_1|+\cdots+k_{in}|u_n|$$

并且设$\max\{\varPhi_1,\varPhi_2,\cdots,\varPhi_n\}<1$, 其中，$\varPhi_i$是由式(10)给出的，则边值问题(1)在[0, 1]上至少存在 1 个解。

证明 由假设条件可知，任意

$$\boldsymbol{U}(t)=(u_1(t),u_2(t),\cdots,u_n(t))^{\mathrm{T}}\in E,$$

$$|f_i(t,u_1(t),u_2(t),\cdots,u_n(t))|\leqslant k_{i0}+k_{i1}|u_1(t)|+k_{i2}|u_2(t)|+\cdots+k_{in}|u_n(t)|=k_{i0}+t^{\alpha-2}(k_{i1}t^{2-\alpha}|u_1(t)|+k_{i2}t^{2-\alpha}|u_2(t)|+\cdots+k_{in}t^{2-\alpha}|u_n(t)|)=k_{i0}+t^{\alpha-2}(k_{i1}\|u_1\|+k_{i2}\|u_2\|+\cdots+k_{in}\|u_n\|) \tag{11}$$

a. 证明算子 T 是全连续。

(a) T是连续算子。

设$\{\boldsymbol{U}_m\}\subset E,\boldsymbol{U}\in E$，当$m\to\infty$时，$\|\boldsymbol{U}_m-\boldsymbol{U}\|_E\to0$，则存在一个常数$\sigma>0$，使得$\|\boldsymbol{U}_m\|_E\leqslant\sigma,\|\boldsymbol{U}\|_E\leqslant\sigma$，即对任意的$t\in[0,1]$，有$|t^{2-\alpha}u_{im}|\leqslant\sigma$，$|t^{2-\alpha}u_i|\leqslant\sigma$。因此，对几乎处处的$t\in[0,1]$，有

$$\lim_{m\to\infty}f_i(t,\boldsymbol{U}_m(t))=\lim_{m\to\infty}f_i(t,t^{\alpha-2}t^{2-\alpha}\boldsymbol{U}_m(t))=f_i(t,t^{\alpha-2}t^{2-\alpha}\boldsymbol{U}(t))=f_i(t,\boldsymbol{U}(t))$$

由引理 6 和式(11)可知，

$$|g_{i1}(t-\tau)(f_i(t,\boldsymbol{U}_m(t))-f_i(t,\boldsymbol{U}(t)))|\leqslant 2\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\left(k_{i0}+t^{\alpha-2}(k_{i1}\|u_1\|+\cdots+k_{in}\|u_n\|)\right)$$

由 Lebesgue 控制收敛定理可得

$$\lim_{m\to\infty}\|T_i\boldsymbol{U}_m-T_i\boldsymbol{U}\|=\lim_{m\to\infty}\sup_{t\in(0,1]}t^{2-\alpha}|T_i\boldsymbol{U}_m-T_i\boldsymbol{U}|=\lim_{m\to\infty}\sup_{t\in(0,1]}t^{2-\alpha}\left(\int_0^t g_{i1}(t-\tau)|f_i(\tau,\boldsymbol{U}_m(\tau))-f_i(\tau,\boldsymbol{U}(\tau))|\mathrm{d}\tau+\frac{g_{i2}(1)g_{i3}(t)-g_{i3}(1)g_{i2}(t)}{\varDelta_i}\int_0^{\eta_i}g_{i1}(\eta_i-\tau)|f_i(\tau,\boldsymbol{U}_m(\tau))-f_i(\tau,\boldsymbol{U}(\tau))|\mathrm{d}\tau+\frac{g_{i3}(\eta_i)g_{i2}(t)-g_{i2}(\eta_i)g_{i3}(t)}{\varDelta_i}\cdot\int_0^1 g_{i1}(1-\tau)|f_i(\tau,\boldsymbol{U}_m(\tau))-f_i(\tau,\boldsymbol{U}(\tau))|\mathrm{d}\tau\right)=0$$

故$\lim\limits_{m\to\infty}\|T\boldsymbol{U}_m-T\boldsymbol{U}\|_E\to0$，则算子$T$是连续算子。

(b) T是相对列紧。

首先证明算子T在 E 上一致有界。令$\varOmega\subset E$有界，对任意$\boldsymbol{U}\in\varOmega$，存在$M>0$，使得$\|\boldsymbol{U}\|_\varOmega\leqslant M$。由引理 6 可知，

$$|g_{11}(t)|<\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_1)}$$

由式(11)可得

$$|f_i(t,\boldsymbol{U}(t))|\leqslant k_{i0}+t^{\alpha-2}(k_{i1}\|u_1\|+k_{i2}\|u_2\|+\cdots+k_{in}\|u_n\|)\leqslant k_{i0}+Mt^{\alpha-2}\max\{k_{i1},k_{i2},\cdots,k_{in}\} \tag{12}$$

对任意$\boldsymbol{U}\in\varOmega,t\in[0,1]$，结合式(12)，有

$$|t^{2-\alpha}T_i(\boldsymbol{U}(t))|\leqslant\int_0^t|g_{i1}(t-\tau)f_i(\tau,\boldsymbol{U}(\tau))|\mathrm{d}\tau+\left|t^{2-\alpha}\frac{g_{i2}(1)g_{i3}(t)-g_{i3}(1)g_{i2}(t)}{\varDelta_i}\right|\int_0^{\eta_i}|g_{i1}(\eta_i-\tau)f_i(\tau,\boldsymbol{U}(\tau))|\mathrm{d}\tau+\left|t^{2-\alpha}\frac{g_{i3}(\eta_i)g_{i2}(t)-g_{i2}(\eta_i)g_{i3}(t)}{\varDelta_i}\right|\int_0^1|g_{i1}(1-\tau)f_i(\tau,\boldsymbol{U}(\tau))|\mathrm{d}\tau\leqslant\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\left(\int_0^1(k_{i0}+M\tau^{\alpha-2}\max\{k_{i1},k_{i2},\cdots,k_{in}\})\mathrm{d}\tau+l_{i1}\int_0^{\eta_i}(k_{i0}+M\tau^{\alpha-2}\max\{k_{i1},k_{i2},\cdots,k_{in}\})\mathrm{d}\tau+l_{i2}\int_0^1(k_{i0}+M\tau^{\alpha-2}\max\{k_{i1},k_{i2},\cdots,k_{in}\})\mathrm{d}\tau\right)\leqslant(1+\eta_il_{i1}+l_{i2})\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\left(k_{i0}+\frac{M}{\alpha-1}\max\{k_{i1},k_{i2},\cdots,k_{in}\}\right)$$

所以，对任意$\boldsymbol{U}\in\varOmega,t\in[0,1]$，有

$$\|T_i(\boldsymbol{U})\|\leqslant(1+\eta_il_{i1}+l_{i2})\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\cdot\left(k_{i0}+\frac{M}{\alpha-1}\max\{k_{i1},k_{i2},\cdots,k_{in}\}\right) \tag{13}$$

由不等式(13)可得

$$\|T(\boldsymbol{U})\|_E=\sum_{i=1}^n\|T_i(\boldsymbol{U})\|\leqslant\sum_{i=1}^n(1+\eta_il_{i1}+l_{i2})\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\cdot\left(k_{i0}+\frac{M}{\alpha-1}\max\{k_{i1},k_{i2},\cdots,k_{in}\}\right)$$

因此,算子T一致有界。

然后再证算子T等度连续。只需证明算子T_i等度连续。

函数 $g_{i1}(t)$, $t^{2-\alpha}g_{i2}(t)$, $t^{2-\alpha}g_{i3}(t)$在 $[0,1]$上连续，所以，函数$g_{i1}(t)$, $t^{2-\alpha}g_{i2}(t)$, $t^{2-\alpha}g_{i3}(t)$在$[0,1]$上一致连续，即对任意$\varepsilon>0$，存在常数$\delta>0$，当$t_1,t_2\in[0,1]$，$t_1<t_2$，$|t_2-t_1|<\delta$时，都有

$$|g_{i1}(t_2-\tau)-g_{i1}(t_1-\tau)|<\frac{\varepsilon}{5(k_{i0}+M(\alpha-1)^{-1}\max\{k_{i1},\cdots,k_{in}\})}$$

$$|t_2^{\alpha-1}-t_1^{\alpha-1}|<\frac{(\alpha-1)\varepsilon}{5(\Gamma^{-1}(\alpha)+\mathrm{e}^{(1-\lambda_i)})M\max\{k_{i1},\cdots,k_{in}\}}$$

$$|t_2^{2-\alpha}g_{i2}(t_2)-t_1^{2-\alpha}g_{i2}(t_1)|<\frac{|\varDelta_i|(\eta_i|g_{i3}(1)|+|g_{i3}(\eta_i)|)^{-1}\varepsilon}{5(\Gamma^{-1}(\alpha)+\mathrm{e}^{(1-\lambda_i)})(k_{i0}+M(\alpha-1)^{-1}\max\{k_{i1},\cdots,k_{in}\})}$$

$$|t_2^{2-\alpha}g_{i3}(t_2)-t_1^{2-\alpha}g_{i3}(t_1)|<\frac{|\varDelta_i|(\eta_i|g_{i2}(1)|+|g_{i2}(\eta_i)|)^{-1}\varepsilon}{5(\Gamma^{-1}(\alpha)+\mathrm{e}^{(1-\lambda_i)})(k_{i0}+M(\alpha-1)^{-1}\max\{k_{i1},\cdots,k_{in}\})}$$

所以，对上述的$\varepsilon>0$，存在常数$\delta\in(0,\varepsilon(5k_{i0}(\Gamma^{-1}(\alpha)+\mathrm{e}^{(1-\lambda_i)}))^{-1})$, 当$t_1,t_2\in[0,1]$, $t_1<t_2$, $|t_2-t_1|<\delta$，则有

$$\begin{aligned}
&|t_2^{2-\alpha}T_i(\boldsymbol{U})(t_2)-t_1^{2-\alpha}T_i(\boldsymbol{U})(t_1)|\leqslant\int_0^{t_1}|(g_{i1}(t_2-\tau)-g_{i1}(t_1-\tau))f_i(\tau,\boldsymbol{U}(\tau))|\mathrm{d}\tau+\\
&\int_{t_1}^{t_2}|g_{i1}(t_2-\tau)f_i(\tau,\boldsymbol{U}(\tau))|\mathrm{d}\tau+|g_{i2}(1)(t_2^{2-\alpha}g_{i3}(t_2)-t_1^{2-\alpha}g_{i3}(t_1))-g_{i3}(1)(t_2^{2-\alpha}g_{i2}(t_2)-t_1^{2-\alpha}g_{i2}(t_1))|\cdot\\
&\frac{1}{|\varDelta_i|}\int_0^{\eta_i}|g_{i1}(\eta_i-\tau)f_i(\tau,\boldsymbol{U}(\tau))|\mathrm{d}\tau+|g_{i3}(\eta_i)(t_2^{2-\alpha}g_{i2}(t_2)-t_1^{2-\alpha}g_{i2}(t_1))-g_{i2}(\eta_i)(t_2^{2-\alpha}g_{i3}(t_2)-t_1^{2-\alpha}g_{i3}(t_1))|\cdot\\
&\frac{1}{|\varDelta_i|}\int_0^{1}|g_{i1}(1-\tau)f_i(\tau,\boldsymbol{U}(\tau))|\mathrm{d}\tau\leqslant\int_0^{1}|g_{i1}(t_2-\tau)-g_{i1}(t_1-\tau)|(k_{i0}+M\tau^{\alpha-1}\max\{k_{i1},\cdots,k_{in}\})\mathrm{d}\tau+\\
&\int_{t_1}^{t_2}|g_{i1}(t_2-\tau)|(k_{i0}+M\tau^{\alpha-1}\max\{k_{i1},\cdots,k_{in}\})\mathrm{d}\tau+\\
&\left(|g_{i2}(1)(t_2^{2-\alpha}g_{i3}(t_2)-t_1^{2-\alpha}g_{i3}(t_1))|+|g_{i3}(1)(t_2^{2-\alpha}g_{i2}(t_2)-t_1^{2-\alpha}g_{i2}(t_1))|\right)\cdot\\
&\frac{1}{|\varDelta_i|}\int_0^{\eta_i}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)(k_{i0}+M\tau^{\alpha-1}\max\{k_{i1},\cdots,k_{in}\})\mathrm{d}\tau+\\
&\left(|g_{i3}(\eta_i)(t_2^{2-\alpha}g_{i2}(t_2)-t_1^{2-\alpha}g_{i2}(t_1))|+|g_{i2}(\eta_i)(t_2^{2-\alpha}g_{i3}(t_2)-t_1^{2-\alpha}g_{i3}(t_1))|\right)\cdot\\
&\frac{1}{|\varDelta_i|}\int_0^{1}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)(k_{i0}+M\tau^{\alpha-1}\max\{k_{i1},\cdots,k_{in}\})\mathrm{d}\tau\leqslant\frac{1}{5}\varepsilon+\frac{1}{5}\varepsilon+\frac{1}{5}\varepsilon+\frac{1}{5}\varepsilon+\frac{1}{5}\varepsilon=\varepsilon
\end{aligned}$$

因此，算子T等度连续。由 Arzela-Ascoli 定理可知，算子T相对列紧。又因为算子T是连续算子，所以，算子T是全连续的。

b. 记$V(T)=\{\boldsymbol{U}\in E:\boldsymbol{U}=\mu T\boldsymbol{U},0<\mu<1\}$，令$\boldsymbol{U}\in V(T)$，则$\boldsymbol{U}=\mu T\boldsymbol{U}$。对任意$t\in[0,1]$，有$u_i(t)=\mu T_i(\boldsymbol{U}(t))$。

由式(8)和式(11)可知，

$$\begin{aligned}
&|t^{2-\alpha}u_i(t)|=|t^{2-\alpha}\mu T_i(\boldsymbol{U}(t))|\leqslant\int_0^{t}|g_{i1}(t-\tau)f_i(\tau,\boldsymbol{U}(\tau))|\,\mathrm{d}\tau+\\
&\left|t^{2-\alpha}\frac{g_{i2}(1)g_{i3}(t)-g_{i3}(1)g_{12}(t)}{\varDelta_i}\right|\int_0^{\eta_i}|g_{i1}(\eta_i-\tau)f_i(\tau,\boldsymbol{U}(\tau))|\,\mathrm{d}\tau+\\
&\left|t^{2-\alpha}\frac{g_{i3}(\eta_i)g_{i2}(t)-g_{i2}(\eta_i)g_{i3}(t)}{\varDelta_i}\right|\int_0^{1}|g_{i1}(1-\tau)f_i(\tau,\boldsymbol{U}(\tau))|\,\mathrm{d}\tau\leqslant\\
&\int_0^{1}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\left(k_{i0}+\tau^{\alpha-2}(k_{i1}\|u_1\|+k_{i2}\|u_2\|+\cdots+k_{in}\|u_n\|)\right)\mathrm{d}\tau+\\
&l_{i1}\int_0^{\eta_i}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\left(k_{i0}+\tau^{\alpha-2}(k_{i1}\|u_1\|+k_{i2}\|u_2\|+\cdots+k_{in}\|u_n\|)\right)\mathrm{d}\tau+\\
&l_{i2}\int_0^{1}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\left(k_{i0}+\tau^{\alpha-2}(k_{i1}\|u_1\|+k_{i2}\|u_2\|+\cdots+k_{in}\|u_n\|)\right)\mathrm{d}\tau=\\
&\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\left(k_{i0}(1+\eta_il_{i1}+l_{i2})+\frac{k_{i1}}{\alpha-1}(1+\eta_il_{i1}+l_{i2})\|u_1\|+\cdots+\frac{k_{in}}{\alpha-1}(1+\eta_il_{i1}+l_{i2})\|u_n\|\right)=\\
&\gamma_i\theta_i\left(k_{i0}+\frac{k_{i1}}{\alpha-1}\|u_1\|+\cdots+\frac{k_{in}}{\alpha-1}\|u_n\|\right)
\end{aligned}$$

因此

$$\|u_i\| \leqslant \gamma_i\theta_i\left(k_{i0}+\frac{k_{i1}}{\alpha-1}\|u_1\|+\cdots+\frac{k_{in}}{\alpha-1}\|u_n\|\right) \quad (14)$$

由式(14)可得

$$\begin{aligned}&\|u_1\|+\|u_2\|+\cdots+\|u_n\| \leqslant \\ &(\gamma_1\theta_1 k_{10}+\gamma_2\theta_2 k_{20}+\cdots+\gamma_n\theta_n k_{n0})+ \\ &\frac{1}{\alpha-1}(\gamma_1\theta_1 k_{11}+\gamma_2\theta_2 k_{21}+\cdots+\gamma_n\theta_n k_{n1})\|u_1\|+\cdots+ \\ &\frac{1}{\alpha-1}(\gamma_1\theta_1 k_{1n}+\gamma_2\theta_2 k_{2n}+\cdots+\gamma_n\theta_n k_{nn})\|u_n\| = \\ &\varPhi_0+\max\{\varPhi_1,\varPhi_2,\cdots,\varPhi_n\}(\|u_1\|+\cdots+\|u_n\|)\end{aligned}$$

所以,

$$\|\boldsymbol{U}\|_E=\|u_1\|+\cdots+\|u_n\| \leqslant \frac{\varPhi_0}{1-\max\ \{\varPhi_1,\varPhi_2,\cdots,\varPhi_n\}}$$

即证得集合V是有界集。因此，由引理 9 可知，算子T至少存在 1 个不动点。所以，边值问题(1)至少存在 1 个解。

证毕。

接下来证明边值问题(1)解的唯一性，采用 Banach 压缩映射原理。

为了证明的方便，引入一些记号：

$$\omega_i=\frac{1}{\alpha-1}(\gamma_1\theta_1 N_{1i}+\gamma_2\theta_2 N_{2i}+\cdots+\gamma_n\theta_n N_{ni}) \quad (15)$$

$$\xi_i=\frac{\gamma_i\theta_i}{\alpha-1}(N_{i1}+N_{i2}+\cdots+N_{in}) \quad (16)$$

定理 2 令 $f_i:[0,1]\times\mathbb{R}^n\to\mathbb{R}$是连续函数，设$\varDelta_i\neq 0$，存在实数$N_{ij}\geqslant 0$，$i,j=1,2,\cdots,n$, 对任意$t\in[0,1], u_i(t), v_i(t)\in E$，使得

$$|f_i(t,u_1,\cdots,u_n)-f_i(t,v_1,\cdots,v_n)| \leqslant \sum_{j=1}^{n} N_{ij}|u_j-v_j|$$

设$\omega_1+\omega_2+\cdots+\omega_n<1$，其中，$\omega_i$是由式(15)定义的，则有：

a. 边值问题(1)在[0, 1]上有唯一解$\boldsymbol{U}^*(t)$;

b. 对任意初始点$\boldsymbol{U}_0\in E$，令

$$\begin{aligned}&\boldsymbol{U}_1=T\boldsymbol{U}_0, \boldsymbol{U}_2=T\boldsymbol{U}_1= \\ &T^2\boldsymbol{U}_0,\cdots,\boldsymbol{U}_m=T\boldsymbol{U}_{m-1}=T^m\boldsymbol{U}_0\end{aligned} \quad (17)$$

边值问题(1)的解$\boldsymbol{U}^*(t)=\lim\limits_{m\to\infty}\boldsymbol{U}_m(t)$, 有误差估计

$$\|\boldsymbol{U}^*-\boldsymbol{U}_m\|_E \leqslant \frac{(\omega_1+\omega_2+\cdots+\omega_n)^m}{1-(\omega_1+\omega_2+\cdots+\omega_n)}\|\boldsymbol{U}_1-\boldsymbol{U}_0\|_E$$

证明 设 $\sup\limits_{t\in(0,1]}|f_i(t,0,0,\cdots,0)|=\rho_i$，使得

$$r\geqslant\max\left\{\frac{n\gamma_1\theta_1\rho_1}{1-n\xi_1},\frac{n\gamma_2\theta_2\rho_2}{1-n\xi_2},\cdots,\frac{n\gamma_n\theta_n\rho_n}{1-n\xi_n}\right\}$$

先证$TB_r\subset B_r$，其中，$B_r=\{U\in E:\|U\|_E\leqslant r\}$。对任意$\boldsymbol{U}(t)\in B_r$，则有

$$\begin{aligned}&|t^{2-\alpha}T_i(u_1(t),\cdots,u_n(t))| \leqslant \int_0^t|g_{i1}(t-\tau)|(|f_i(\tau,u_1(\tau),\cdots,u_n(\tau))-f_i(\tau,0,\cdots,0)|+|f_i(\tau,0,\cdots,0)|)\,\mathrm{d}\tau+ \\ &\left|t^{2-\alpha}\frac{g_{i2}(1)g_{i3}(t)-g_{i3}(1)g_{i2}(t)}{\varDelta_i}\right|\int_0^{\eta_i}|g_{i1}(\eta_i-\tau)(f_i(\tau,u_1(\tau),\cdots,u_n(\tau))-f_i(\tau,0,\cdots,0))|\mathrm{d}\tau+ \\ &\left|t^{2-\alpha}\frac{g_{i2}(1)g_{i3}(t)-g_{i3}(1)g_{i2}(t)}{\varDelta_i}\right|\int_0^{\eta_i}|g_{i1}(\eta_i-\tau)f_i(\tau,0,\cdots,0)|\mathrm{d}\tau+ \\ &\left|t^{2-\alpha}\frac{g_{i3}(\eta_i)g_{i2}(t)-g_{i2}(\eta_i)g_{i3}(t)}{\varDelta_i}\right|\int_0^1|g_{i1}(1-\tau)(f_i(\tau,u_1(\tau),\cdots,u_n(\tau))-f_i(\tau,0,\cdots,0))|\mathrm{d}\tau+ \\ &\left|t^{2-\alpha}\frac{g_{i3}(\eta_i)g_{i2}(t)-g_{i2}(\eta_i)g_{i3}(t)}{\varDelta_i}\right|\int_0^1|g_{i1}(1-\tau)f_i(\tau,0,\cdots,0)|\mathrm{d}\tau \leqslant \rho_i\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)(1+\eta_i l_{i1}+l_{i2})+ \\ &\int_0^t\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\left(\sum_{j=1}^{n}N_{ij}|u_j(\tau)|\right)\mathrm{d}\tau+l_{i1}\int_0^{\eta_i}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\left(\sum_{j=1}^{n}N_{ij}|u_j(\tau)|\right)\mathrm{d}\tau+l_{i2}\int_0^1\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\left(\sum_{j=1}^{n}N_{ij}|u_j(\tau)|\right)\mathrm{d}\tau \leqslant \\ &\rho_i\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)(1+\eta_i l_{i1}+l_{i2})+\int_0^1\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\tau^{\alpha-2}\left(\sum_{j=1}^{n}N_{ij}\tau^{2-\alpha}|u_j(\tau)|\right)\mathrm{d}\tau+ \\ &l_{i1}\int_0^{\eta_i}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\tau^{\alpha-2}\left(\sum_{j=1}^{n}N_{ij}\tau^{2-\alpha}|u_j(\tau)|\right)\mathrm{d}\tau+l_{i2}\int_0^1\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\tau^{\alpha-2}\left(\sum_{j=1}^{n}N_{ij}\tau^{2-\alpha}|u_j(\tau)|\right)\mathrm{d}\tau \leqslant \\ &\rho_i\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)(1+\eta_i l_{i1}+l_{i2})+\frac{1}{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\left(\sum_{j=1}^{n}N_{ij}\|u_j\|\right)+ \\ &\frac{\eta_i l_{i1}}{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\left(\sum_{j=1}^{n}N_{ij}\|u_j\|\right)+\frac{l_{i2}}{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\left(\sum_{j=1}^{n}N_{ij}\|u_j\|\right) \leqslant \\ &\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)(1+\eta_i l_{i1}+l_{i2})\left(\frac{1}{\alpha-1}(N_{i1}+N_{i2}+\cdots+N_{in})r+\rho_i\right)=\gamma_i\theta_i\left(\frac{1}{\alpha-1}(N_{i1}+N_{i2}+\cdots+N_{in})r+\rho_i\right)\end{aligned}$$

因此,

$$\|T_i(\boldsymbol{U})\| \leqslant \gamma_i\theta_i\left(\frac{1}{\alpha-1}(N_{i1}+N_{i2}+\cdots+N_{in})r+\rho_i\right) \leqslant \frac{r}{n}$$

即

$$\|T(\boldsymbol{U})\|_E = \left|\sum_{i=1}^{n} T_i(\boldsymbol{U})\right| \leqslant \sum_{i=1}^{n}\gamma_i\theta_i\left(\frac{1}{\alpha-1}(N_{i1}+N_{i2}+\cdots+N_{in})r+\rho_i\right) \leqslant \frac{r}{n}+\cdots+\frac{r}{n}=r$$

所以，$TB_r \subset B_r$。

接下来证明算子T是压缩的。对任意$t\in[0,1]$, $\boldsymbol{U}(t)=(u_1(t),\cdots,u_n(t))^{\mathrm{T}}\in E, \boldsymbol{V}(t)=(v_1(t),\cdots,v_n(t))^{\mathrm{T}}\in E$，则有

$$\begin{aligned}
&|t^{2-\alpha}T_i(\boldsymbol{U})(t)-t^{2-\alpha}T_i(\boldsymbol{V})(t)| \leqslant \int_0^t |g_{i1}(t-\tau)(f_i(\tau,\boldsymbol{U}(\tau))-f_i(\tau,\boldsymbol{V}(\tau)))|\mathrm{d}\tau+\\
&\left|t^{2-\alpha}\frac{g_{i2}(1)g_{i3}(t)-g_{i3}(1)g_{i2}(t)}{\varDelta_i}\right|\int_0^{\eta_i}|g_{i1}(\eta_i-\tau)(f_i(\tau,\boldsymbol{U}(\tau))-f_i(\tau,\boldsymbol{V}(\tau)))|\mathrm{d}\tau+\\
&\left|t^{2-\alpha}\frac{g_{i3}(\eta_i)g_{i2}(t)-g_{i2}(\eta_i)g_{i3}(t)}{\varDelta_i}\right|\int_0^{1}|g_{i1}(1-\tau)(f_i(\tau,\boldsymbol{U}(\tau))-f_i(\tau,\boldsymbol{V}(\tau)))|\mathrm{d}\tau \leqslant\\
&\int_0^1\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\tau^{\alpha-2}\left(\sum_{j=1}^{n}N_{ij}\tau^{2-\alpha}|u_j-v_j|\right)\mathrm{d}\tau+l_{i1}\int_0^{\eta_i}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\tau^{\alpha-2}\left(\sum_{j=1}^{n}N_{ij}\tau^{2-\alpha}|u_j-v_j|\right)\mathrm{d}\tau+\\
&l_{i2}\int_0^1\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)\tau^{\alpha-2}\left(\sum_{j=1}^{n}N_{ij}\tau^{2-\alpha}|u_j-v_j|\right)\mathrm{d}\tau \leqslant\\
&\frac{1}{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)(N_{i1}\|u_1-v_1\|+N_{i2}\|u_2-v_2\|+\cdots+N_{in}\|u_n-v_n\|)+\\
&\frac{\eta_i l_{i1}}{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)(N_{i1}\|u_1-v_1\|+N_{i2}\|u_2-v_2\|+\cdots+N_{in}\|u_n-v_n\|)+\\
&\frac{l_{i2}}{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{e}^{(1-\lambda_i)}\right)(N_{i1}\|u_1-v_1\|+N_{i2}\|u_2-v_2\|+\cdots+N_{in}\|u_n-v_n\|)=\\
&\frac{\gamma_i\theta_i}{\alpha-1}(N_{i1}\|u_1-v_1\|+N_{i2}\|u_2-v_2\|+\cdots+N_{in}\|u_n-v_n\|)
\end{aligned}$$

因此，对任意$t\in[0,1], \boldsymbol{U}(t)=(u_1(t),\cdots,u_n(t))^{\mathrm{T}}, \boldsymbol{V}(t)=(v_1(t),\cdots,v_n(t))^{\mathrm{T}}\in E$，则有

$$|t^{2-\alpha}T_i(\boldsymbol{U})(t)-t^{2-\alpha}T_i(\boldsymbol{V})(t)| \leqslant \frac{\gamma_i\theta_i}{\alpha-1}(N_{i1}\|u_1-v_1\|+N_{i2}\|u_2-v_2\|+\cdots+N_{in}\|u_n-v_n\|) \tag{18}$$

由式(18)可知，

$$\begin{aligned}
&\|T(\boldsymbol{U})-T(\boldsymbol{V})\|_E \leqslant \frac{1}{\alpha-1}(\gamma_1\theta_1N_{11}+\gamma_2\theta_2N_{21}+\cdots+\gamma_n\theta_nN_{n1})\|u_1-u_1\|+\cdots+\\
&\frac{1}{\alpha-1}(\gamma_1\theta_1N_{1n}+\gamma_2\theta_2N_{2n}+\cdots+\gamma_n\theta_nN_{nn})\|u_n-v_n\| \leqslant (\omega_1+\cdots+\omega_n)(\|u_1-v_1\|+\cdots+\|u_n-v_n\|)
\end{aligned} \tag{19}$$

因为，$\omega_1+\omega_2+\cdots+\omega_n<1$，所以，算子$T$是压缩算子。由 Banach 压缩映射定理可知，算子T有唯一的不动点，所以，边值问题(1)在$t\in[0,1]$上有唯一解$\boldsymbol{U}^*(t)$。对任意的$k>m$，有

$$\begin{aligned}
&\|\boldsymbol{U}_m-\boldsymbol{U}_k\|_E \leqslant \|\boldsymbol{U}_m-\boldsymbol{U}_{m+1}\|_E+\|\boldsymbol{U}_{m+1}-\boldsymbol{U}_{m+2}\|_E+\cdots+\|\boldsymbol{U}_{k-1}-\boldsymbol{U}_k\|_E=\\
&\|T\boldsymbol{U}_{m-1}-T\boldsymbol{U}_m\|_E+\|T\boldsymbol{U}_m-T\boldsymbol{U}_{m+1}\|_E+\cdots+\|T\boldsymbol{U}_{k-2}-T\boldsymbol{U}_{k-1}\|_E \leqslant\\
&\left((\omega_1+\omega_2+\cdots+\omega_n)^m+(\omega_1+\omega_2+\cdots+\omega_n)^{m+1}+\cdots+(\omega_1+\omega_2+\cdots+\omega_n)^{k-1}\right)\|\boldsymbol{U}_1-\boldsymbol{U}_0\|_E=\\
&(\omega_1+\omega_2+\cdots+\omega_n)^m\left(1+(\omega_1+\omega_2+\cdots+\omega_n)+\cdots+(\omega_1+\omega_2+\cdots+\omega_n)^{k-m-1}\right)\|\boldsymbol{U}_1-\boldsymbol{U}_0\|_E=\\
&(\omega_1+\omega_2+\cdots+\omega_n)^m\frac{1-(\omega_1+\omega_2+\cdots+\omega_n)^{k-m}}{1-(\omega_1+\omega_2+\cdots+\omega_n)}\|\boldsymbol{U}_1-\boldsymbol{U}_0\|_E \leqslant \frac{(\omega_1+\omega_2+\cdots+\omega_n)^m}{1-(\omega_1+\omega_2+\cdots+\omega_n)}\|\boldsymbol{U}_1-\boldsymbol{U}_0\|_E
\end{aligned}$$

又因为，$\omega_1+\omega_2+\cdots+\omega_n<1$，得到迭代序列$\{\boldsymbol{U}_m\}$收敛，所以，存在极限$\bar{\boldsymbol{U}}(t)=\lim\limits_{m\to\infty}\boldsymbol{U}_m(t)$。

由三角不等式和式(19)可得

$$\|\bar{\boldsymbol{U}}-T\bar{\boldsymbol{U}}\|_E\leqslant\|\bar{\boldsymbol{U}}-\boldsymbol{U}_m\|_E+\|\boldsymbol{U}_m-T\bar{\boldsymbol{U}}\|_E\leqslant$$
$$\|\bar{\boldsymbol{U}}-\boldsymbol{U}_m\|_E+(\omega_1+\omega_2+\cdots+\omega_n)\|\boldsymbol{U}_{m-1}-\bar{\boldsymbol{U}}\|_E$$

因为，$\bar{\boldsymbol{U}}(t)=\lim\limits_{m\to\infty}\boldsymbol{U}_m$，所以，当$m\to\infty$时，

$$\|\bar{\boldsymbol{U}}-\boldsymbol{U}_m\|_E+(\omega_1+\omega_2+\cdots+\omega_n)\|\boldsymbol{U}_{m-1}-\bar{\boldsymbol{U}}\|_E\to 0$$

得到$\|\bar{\boldsymbol{U}}-T\bar{\boldsymbol{U}}\|_E=0$，即$\bar{\boldsymbol{U}}=T\bar{\boldsymbol{U}}$。由定理 2 可知，边值问题(1)存在唯一解$\boldsymbol{U}^*$，所以，

$$\boldsymbol{U}^*=\bar{\boldsymbol{U}}$$

迭代序列式(17)对任意的$\boldsymbol{U}_0\in E$都收敛到 T 的唯一不动点$\boldsymbol{U}^*$，且有误差估计

$$\|\boldsymbol{U}^*-\boldsymbol{U}_m\|_E\leqslant\frac{(\omega_1+\omega_2+\cdots+\omega_n)^m}{1-(\omega_1+\omega_2+\cdots+\omega_n)}\|\boldsymbol{U}_1-\boldsymbol{U}_0\|_E$$

证毕。

4 例 题

为了说明所得结论具有较好的适用性，考虑几个具体的问题。

例 1 考虑分数阶微分方程组边值问题

$$\begin{cases}\mathrm{D}^\alpha u_1(t)+0.99\mathrm{D}^{\alpha-1}u_1(t)=f_1(t,u_1(t),v_1(t)),\\ \quad 0<t<1\\ \mathrm{D}^\alpha u_2(t)+0.98\mathrm{D}^{\alpha-1}u_2(t)=f_2(t,u_1(t),v_1(t)),\\ \quad 0<t<1\\ u_1(\eta_1)=u_1(1)=0\\ u_2(\eta_2)=u_2(1)=0\end{cases}\tag{20}$$

其中，$\alpha=\frac{3}{2}$, $\lambda_1=0.99$, $\lambda_2=0.98$是(0, 1)上的 2 个正实数，取$\eta_1=\frac{1}{2}$, $\eta_2=\frac{1}{4}$。

通过计算。得到

$$\begin{cases}\varDelta_1\approx-0.321\,394\neq0,\ \varDelta_2\approx-0.235\,721\neq0\\ \gamma_1\approx2.138\,429,\gamma_2\approx2.148\,581\\ l_{11}\approx2.133\,16,l_{12}\approx0.572\,013\\ l_{21}\approx2.026\,437,l_{22}\approx2.276\,976\\ \theta_1\approx4.093\,02,\theta_2\approx3.419\,979\end{cases}\tag{21}$$

令

$$f_1(t,u_1,u_2)=\frac{t^2}{900}\left(1+\frac{1}{789}u_1+\frac{1}{596}\sin u_2\right),$$
$$t\in[0,1],u_1,u_2\in\mathbb{R}$$
$$f_2(t,u_1,u_2)=\frac{t}{999}\left(2+\frac{1}{900}\sin u_1+\frac{1}{127}\cos^2 u_2\right),$$
$$t\in[0,1],u_1,u_2\in\mathbb{R}$$

即

$$|f_1(t,u_1,u_2)|=\left|\frac{t^2}{900}\left(1+\frac{1}{789}u_1+\frac{1}{596}\sin u_2\right)\right|\leqslant$$
$$1+\frac{1}{789}|u_1|+\frac{1}{1\,200}|u_2|$$
$$|f_2(t,u_1,u_2)|=\left|\frac{t}{999}\left(2+\frac{1}{900}\sin u_1+\frac{1}{127}\cos^2 u_2\right)\right|\leqslant$$
$$2+\frac{1}{900}|u_1|+\frac{1}{137}|u_2|$$

并且, $k_{11}=\frac{1}{789}$，$k_{12}=\frac{1}{1\,200}$，$k_{21}=\frac{1}{900}$，$k_{22}=\frac{1}{137}$。所以，

$$\frac{1}{\alpha-1}(\gamma_1\theta_1k_{11}+\gamma_1\theta_1k_{21})\approx0.041\,636\,9<1$$
$$\frac{1}{\alpha-1}(\gamma_2\theta_2k_{12}+\gamma_2\theta_2k_{22})\approx0.140\,376<1$$

因此，所有的条件都满足定理 1，所以，由定理 1 可知，微分方程组(20)至少存在 1 个解。

例 2 在例 1 的基础上，只改变函数f_1,f_2，其他的条件保持不变，$\varDelta_i,l_{1i},l_{2i},\gamma_i,\theta_i(i=1,2)$是式(20)中的，其中，$\alpha=\frac{3}{2}$。令

$$f_1(t,u_1,u_2)=1+\frac{t^2}{790}u_1+\frac{1}{960}\arctan u_2,$$
$$t\in[0,1],u_1,u_2\in\mathbb{R}$$
$$f_2(t,u_1,u_2)=2+\frac{t}{460}\sin u_1+\frac{5}{239}\cos u_2,t\in[0,1],$$
$$u_1,u_2\in\mathbb{R}\tag{22}$$

即

$$|f_1(t,u_1,u_2)-f_1(t,v_1,v_2)|\leqslant\frac{1}{790}|u_1-v_1|+\frac{1}{960}|u_2-v_2|$$
$$|f_2(t,u_1,u_2)-f_2(t,v_1,v_2)|\leqslant\frac{1}{460}|u_1-v_1|+\frac{5}{239}|u_2-v_2|$$

并且，$N_{11}=\frac{1}{790},N_{12}=\frac{1}{960},N_{21}=\frac{1}{460},N_{22}=\frac{5}{239}$。所以，

$$\frac{1}{\alpha-1}[(\gamma_1\theta_1N_{11}+\gamma_2\theta_2N_{21})+$$
$$(\gamma_1\theta_1N_{12}+\gamma_2\theta_2N_{22})]\approx0.379\,794$$

所有的条件满足定理 2，因此，微分方程组(20)有唯一解 (f_1,f_2是式(21)中给定的)。

对任意$\boldsymbol{U}_0=(0,0)^{\mathrm{T}}$，则$\boldsymbol{U}_1=(0.041\,565, 0.043\,789)^{\mathrm{T}}$，且有误差估计

$$\|\boldsymbol{U}^*-\boldsymbol{U}_{200}\|\leqslant1.272\,61\times10^{-84}\|\boldsymbol{U}_1-\boldsymbol{U}_0\|$$

参考文献:

[1] BALEANU D, DIETHELM K, SCALAS E, et al. Fractional calculus: models and numerical methods[M]//

LUO A C J. Series on Complexity, Nonlinearity and Chaos. Bostin: Word Scientific, 2012: 305–307.

[2] DAS S. Functional fractional calculus for system identification and controls[M]. New York: Springer, 2008.

[3] DJORDJEVIĆ V D, JARIĆ J, FABRY B, et al. Fractional derivatives embody essential features of cell rheological behavior[J]. Annals of Biomedical Engineering, 2003, 31(6): 692–699.

[4] BAI Z B, LÜ H S. Positive solutions for boundary value problem of nonlinear fractional differential equation[J]. Journal of Mathematical Analysis and Applications, 2005, 311(2): 495–505.

[5] 胡雨欣, 寇春海, 葛富东. Banach 空间中分数阶微分方程解的存在性[J]. 东华大学学报 (自然科学版), 2015, 41(6): 867–872.

[6] 吴贵云, 刘锡平, 杨浩. 具有微分算子的分数阶微分方程边值问题解的存在性与唯一性[J]. 上海理工大学学报, 2015, 37(3): 205–209.

[7] LIU X P, JIA M. Existence of solutions for the integral boundary value problems of fractional order impulsive differential equations[J]. Mathematical Methods in the Applied Sciences, 2016, 39(3): 475–487.

[8] 李燕, 刘锡平, 李晓晨, 等. 具逐项分数阶导数的积分边值问题正解的存在性[J]. 上海理工大学学报, 2016, 38(6): 511–516.

[9] LIU X P, LI F F, JIA M, et al. Existence and uniqueness of the solutions for fractional differential equations with nonlinear boundary conditions[J]. Abstract and Applied Analysis, 2014, 2014: 758390.

[10] JIA M, LIU X P. Three nonnegative solutions for fractional differential equations with integral boundary conditions[J]. Computers & Mathematics with Applications, 2011, 62(3): 1405–1412.

[11] ALSAEDI A, ALJOUDI S, AHMAD B. Existence of solutions for Riemann-Liouvillle type coupled systems of fractional integro-differential equations and boundary conditions[J]. Electronic Journal of Differential Equations, 2016, 2016(211): 1–14.

[12] 周志恒, 韦煜明. 半无穷区间上具有共振的序列分数阶微分方程积分边值问题的可解性[J]. 应用数学, 2018, 31(3): 572–582.

[13] MERAL F C, ROYSTON T J, MAGIN R. Fractional calculus in viscoelasticity: an experimental study[J]. Communications in Nonlinear Science and Numerical Simulation, 2010, 15(4): 939–945.

[14] AHMAD B, LUCA R. Existence of solutions for a sequential fractional integro-differential system with coupled integral boundary conditions[J]. Chaos, Solitons & Fractals, 2017, 104: 378–388.

[15] LI Y H, SANG Y B, ZHANG H Y. Solvability of a coupled system of nonlinear fractional differential equations with fractional integral conditions[J]. Journal of Applied Mathematics and Computing, 2016, 50(1/2): 73–91.

[16] 杨巧林. 复变函数与积分变换[M]. 北京: 机械工业出版社, 2007.

[17] KILBAS A A, SRIVASTAVA H M, TRUJILLO J J. Theory and applications of fractional differential equations[M]//VON MILL J. North-Holland Mathematics Studies. Amsterdam: Elsevier, 2006: 1–523.

[18] MCLENNAN A. Fixed point theorems[M]//DURLAUF S N, BLUME L E. The New Palgrave Dictionary of Economics. New York: Springer, 2005.

（编辑：石　瑛）