文章编号:1007-6735(2011)03-0205-24

Supernetworks: The Science of Complexity

Anna Nagurney
(Department of Finance and Operations Management, Isenberg School of Management, University of Massachusetts, Amherst, Massachusetts 01003)

Abstract: This paper provides an overview of the science of supernetworks, highlighting some of the major achievements in terms of theory, methodology, and applications. The goal is to demonstrate the depth and breadth of contributions to the modeling, analysis, and solution of complex networks that underly our modern economies and societies. Some promising areas for future research are also identified.

Key words: *supernetworks*; *networks*; *complex networks*; *complexity*

1 Introduction and Motivation

Science advances through innovations in observation, theory, methodologies, experimentation and data collection. Science builds on earlier discoveries and illuminates our understanding of the world (and beyond) by providing tools to guide and assist in decision-making.

The world has become increasingly complex with the human population soaring to almost 7 billion, with our resources from natural to financial ones being limited, and with the number of disasters growing, as well as the people affected by them. At the same time, the world is more interconnected through communication and transportation technologies. Hence, disruptions in one part of the world may propagate and impact sectors in other parts, even thousands of miles away from the original disruption.

Networks provide the infrastructure for connectivity and for the functioning of our modern economies and societies. Transportation networks give us the means for mobility and the shipment and delivery of goods. Communication networks today allow for the spread of information at speeds never before imagined. Logistical networks enable the manufacture of products and their delivery to points of demand across the globe. The reality of today's networks include: large-scale nature and complexity, increasing congestion, especially in, but not limited to, transportation and telecommunications, alternative behaviors of users of the networks, which can lead to paradoxical phenomena, as well as interactions between the networks themselves (which was rarely observed prior to the advent of the Internet). The decisions made by the users of the networks, in turn, may affect not only the users themselves but others, as well, in terms of profits and costs, the timeliness of deliveries, and the quality of the environment.

Indeed, many of today's networks are characterized by both a large-scale nature and complexity of the underlying network topology. For example, in Chicago's Regional Transportation Network, there are 12 982 nodes, 39 018 links, and 2 297 945 origin/destination (O/D) pairs[1], whereas in the Southern California Association of Governments

Received date: 2011-04-19
Biography: Anna Nagurney, John F. Smith Memorial Professor and Director of the Virtual Center for Supernetworks.

model there are 3 217 origins and/or destinations, 25 428 nodes, and 99 240 links, plus 6 distinct classes of users[2]. In the case of the Internet, in 2010, there were 1.8 billion users[3].

In regards to congestion, in the case of transportation networks in the United States alone, in 2009, it was estimated that congestion resulted in \$115 billion in lost productivity, requiring 4.8 billion hours of extra travel time[4].

Supernetworks are "networks of networks," and their prevalence in the world around us is illustrated by multimodal transportation networks; complex logistical networks consisting of manufacturers, shippers and carriers, distributors, and retailers; electric power generation and distribution networks, multitiered financial networks, and such software technology social network platforms as Facebook and Twitter, along with the Internet itself.

Supernetwork theory, as a body of scientific inquiry, has harnessed optimization theory, network theory, game theory, multicriteria decision-making, the theory of variational inequalities[5], as well as projected dynamical systems theory[6-7], and the references therein, and, in the process, has helped in the advancement and integration, as appropriate, of such methodologies and has broadened both models and applications thereof[8-9].

In particular, the supernetwork framework allows one to formalize the alternatives available to decision-makers, to model their individual behavior, typically, characterized by particular criteria which they wish to optimize, and to, ultimately, compute the flows on the supernetwork, which may consist of product shipments, travelers between origins and destinations, financial flows, information flows, resource and energy flows, as well as the associated costs and "prices." Hence, the concern is with complex decision-making and how the supernetwork concept can be utilized to crystallize and inform in this dimension.

"Super" networks are networks that are "above and beyond" existing classical networks, which consist of nodes, links, and flows, with nodes corresponding to locations in space, links to connections in the form of roads, cables, etc., and flows to vehicles, data, etc. Supernetworks are conceptual in scope, graphical in perspective, and, with the accompanying theory, predictive in nature. Moreover, supernetworks also enable the conceptualization, abstraction, and illumination of a spectrum of problems that may not appear, initially, to involve networks at all, such as a variety of financial and economic problems as well as knowledge production and dissemination. Hence, the study of supernetworks is not limited to physical networks where nodes coincide with locations in space but applies also to abstract networks. The ability to harness the power of a supernetwork formalism provides a competitive advantage since:

(1) many present-day problems are concerned with flows be they, material, human, capital, energy, and/or informational over space and time and, hence, well-suited as an application domain for supernetwork theory;

(2) a graphical or visual depiction of different problems provides a commonality for both theory and practice, and an enhancement of problem description;

(3) one may identify similarities and differences in distinct problems through their underlying network structure, and gain new, deeper insights as a consequence;

(4) one may apply efficient network algorithms for problem solution, since network problems today very often being large-scale.

The origin of supernetworks, as a science for complexity, begins with the invited essay of Nagurney[10] in *OR/MS Today*. In that essay, the need to capture the interrelationships among the foundational networks in our economies and societies was emphasized and it was noted that: "The interactions among transportation networks, telecommunication networks, as well as financial networks is creating supernetworks..." . The essay, in turn, was based on a Distinguished Faculty Lecture given at the University of Massachusetts on April 5, 2000[11]. Subsequent research, and interest, plus funding support, led to the establishment of the Virtual Center for Supernetworks at the Isenberg School of Management at the University of Massachusetts Amherst in the Fall of 2001, followed by the first book on supernetworks by Nagurney and Dong[8].

Below the theme of supernetworks is further elaborated upon and, in particular, the origins of the concept and the term *supernetworks* identified. The goal of this paper is to provide a high-level overview of supernetworks and to highlight both the foundations and some of the successes with a focus on developments since Nagurney and Dong[8].

Transportation, telecommunication, and economic and financial networks have served not only as the basis for the origins of the term "supernetwork," but are also the principal subnetworks in the applications that are relevant to decision-making today. In addition, interestingly, and, as noted in Nagurney and Dong[8], the term "supernetworks" is also used in genetics and biology.

Clearly, transportation networks are complex network systems, in which the decisions of individual travelers affect the efficiency and productivity of the entire system. Transportation networks come in many forms from urban networks and freight networks to airline networks. The "supply" in such a network system is represented by the network topology and the underlying cost characteristics, whereas the "demand" is represented by the users of the network system, that is, the travelers.

Dafermos[12] established how a *multiclass* transportation network could be transformed into a single-class transportation network through the construction of an expanded *abstract* network consisting of as many copies of the original network as there were classes of users (which could, equivalently, correspond to modes of transportation). Moreover, she noted that such networks "arise not only in street networks where vehicles of different types share the same roads (e.g., trucks and passenger cars) but also in other types of transportation networks (e.g., telephone networks)." Therefore, she explicitly recognized that abstract networks could be used to handle multiclass/multimodal transportation networks as well as telecommunication networks! In addition, she considered both user-optimizing and system-optimizing behavior, terms which she had coined with Sparrow in a paper in 1969. Fascinatingly, the concepts of user-optimization versus system-optimization from transportation[13-14] are now being used in the formal study of telecommunication networks, including the Internet[15] with the Braess Daradox[16-17] also being relevant to both application domains[18].

As emphasized in Nagurney and Dong[8], Dafermos[19] proposed an integrated transportation network equilibrium model in which one could formally capture the entire transportation planning process, consisting of origin selection, or destination selection, or both, in addition to route selection, in an optimal way, as path choices over an appropriately constructed *abstract* network. The genesis and formal treatment of decisions more complex than route choices as *path* choices on abstract networks, that is, supernetworks, were, hence, evident as early as 1972 and 1976.

The importance and wider relevance of such abstract networks in decision-making, with a continued focus on transportation planning, were further emphasized through the term "hypernetwork," used by Sheffi[20] and Sheffi and Daganzo[21-22], which was later renamed as a "supernetwork" by Sheffi[23].

As also noted in Nagurney and Dong[8], the recognition and appropriate construction of *abstract* networks was pivotal in that it allowed for the incorporation of transportation-related decisions (where as noted by Dafermos[12], transportation applied also to communication networks) which were not based only on route selection in a classical sense, that is, what route should one take from one's origin, say, place of residence, to one's destination, say, place of employment. Abstract networks, with origins and destinations corresponding to appropriately defined nodes, links connecting nodes having associated disutilities (that is, costs), and paths comprised of links (directed) connecting the origins and destinations, could capture such travel alternatives as not simply just a route but, also, the "mode" of travel, that is, for example, whether one chose to use private or public transportation. Moreover, through the use of both added abstract links and paths, and abstract origin and destination nodes one could include the selection of such locational decisions as the origins and destinations themselves within the same decision-making framework.

The behavioral principle utilized by Dafermos[19] and by Sheffi and Daganzo[21-22] (who also formulated stochastic models) was that decision-makers select the "cost-minimizing" routes among all their available choices. Hence, they behaved according to Wardrop's[24] first principle of travel behavior, which corresponds to what is referred to now as user-optimization, as opposed to system-optimization (Wardrop's second principle of travel behavior). Note that the user-optimization corresponds to decentralized decision-making as opposed to centralized decision-making.

Dafermos[12] considered both system-optimization as well as user-optimization. According to Sheffi and Daganzo[21], the selection of cost-minimizing routes "this is consistent with the principle of utility maximization of choice theory." Moreover, they stated that: "Although hypernetworks enable us to visualize choice problems in a unified way... their main advantage is that they enable us to perform supply-demand equilibrium analysis on a mathematically consistent basis with disaggregate demand models." Additional references to supernetworks and transportation can be found in the book by Nagurney and Dong[8].

We now turn to a discussion of the use of the term "supernetworks" in the context of telecommunication networks. Denning[25] noted that a system of national supercomputer centers would be a network of networks, that is, a "supernetwork", and a powerful tool for science. In addition, he emphasized the importance of location-independent naming, so that if a physical location of a resource would change, none of the supporting programs or files would need to be edited or recompiled. Hence, in a sense, his view of supernetworks is in concert with that of ours in that nodes do not need to correspond to locations in space and may have an abstract association.

Schubert, Goebel, and Cercone[26] used the term "supernetworks" in the context of knowledge representation and stated that: "Networks are compositional: a node in a network can be some other network, and the same subnetwork can be a subnetwork of several larger supernetworks, ..."

Fallows[27] recognized that "The Internet is the supernetwork that links computer networks around the world." In 1997, the Illinois Bar Association defined the Internet as: "a supernetwork of computers that links together individual computers and computer networks located at academic, commercial, government and military sites worldwide, generally by ordinary local telephone lines and long-distance transmission facilities. Communications between computers or individual networks on the Internet are achieved through the use of standard, nonproprietary protocols."

The first instance of an abstract network or supernetwork in the context of economic applications, was due to Quesnay[28], who depicted the circular flow of funds in an economy as a network. Cournot[29] not only seems to have first explicitly stated that a competitive price is determined by the intersection of supply and demand curves, but had done so in the context of two spatially separated markets in which the cost of transporting the good between markets was considered, with a network being implicit. Pigou[30], in turn, studied a transportation network consisting of two routes and noted that the "system-optimized" solution was distinct from the "user-optimized" solution.

Since Quesnay's contribution, numerous economic and financial models have been constructed over abstract networks. In particular, Dafermos and Nagurney[31] identified the isomorphism between transportation network equilibrium problems and spatial price equilibrium problems, whose development had been initiated by Enke[32] and Samuelson[33] (who noted the bipartite network structure of the classical spatial price equilibrium problem), further advanced by Takayama and Judge[34-35], and, subsequently, by Florian and Los[36] and Dafermos and Nagurney[37-38]. The volumes by Nagurney and Siokos[39] and Nagurney[40] contain many models and applications of financial networks, along with references. Nagurney[41] provides an overview of network economics.

A plethora of abstract networks in economics were modeled and studied in the book by Nagurney[5], which also contains extensive references to the subject. In the book on supernetworks, we have demonstrated that the abstract network concept also captures the interactions between/among the underlying networks of economies and societies.

Nagurney et al.[42] demonstrated how multitiered supply chains with electronic commerce could be modeled in an integrated network model, using as the foundation the general supply chain network equilibrium model of Nagurney, Dong and Zhang[43], which was formulated as a variational inequality problem. In addition, Nagurney et al.[44] developed a dynamic multilevel supernetwork model which included financial, informational, and logistical flows using projected dynamical systems. Subsequent supernetwork modeling innovations included the integration of electronic transactions with multitiered financial networks by Nagurney and Ke[45] as well as the introduction of environmental decision-making into multitiered complex supply chain networks by Nagurney and Toyasaki[46]. Moreover, Nagurney, Dong, and Mokhtarian[47] developed a multicriteria supernetwork framework for the quantification of commuting versus telecommuting decision-making as well as shopping versus teleshopping decision-making with multiple decision-makers. Global issues were modeled in the context of supply chains by Nagurney, Cruz, and Matsypura[48] and for financial networks by Nagurney and Cruz[49].

The term *supernetwork* has also been used in biology, specifically, in genetics. Noveen, Hartenstein, and Chuong[50] noted that interacting genes give rise to a gene network, with many interacting gene networks giving rise to a gene "supernetwork." Interestingly, supernetworks are getting increased prominence in biology, notably, in ecology, and even in neuroscience, a topic that we return to in the final section of this paper.

This paper is organized as follows. In Section 2, the fundamental methodologies used in supernetwork modeling and analysis from both static and dynamic perspectives are reviewed. Section 3 describes how supernetwork theory has answered questions posed over half a century ago in the case of finance and electric power. Section 4 then turns to network efficiency and performance assessment, and vulnerability analysis, as initiatited by Nagurney and Qiang[51-53] and shows how, through the unification of a variety of network systems through supernetworks, the importance and ranking of network components can be determined. Section 5 identifies disciplines in which recent breakthroughs have been made because of the methodological advances related to supernetworks and suggests directions for future research.

2 Fundamental Methodologies

In this Section, for completeness, we review some of the fundamental methodologies that have been used for the modeling, analysis, and solution of supernetwork problems. We begin with the theory of variational inequalities and then recall some of the important results for projected dynamical systems. Since the focus is on a scientific perspective for supernetworks, it is important to highlight the methodologies that have had the most impact on supernetwork model formulation, analysis, solution, and application.

In Section 2.1, we discuss the variational inequality problem and in Section 2.2, projected dynamical systems.

2.1 The Variational Inequality Problem

Variational inequalities were introduced by Hartman and Stampacchia[54], principally, for the study of partial differential equation problems drawn from mechanics. That research focused on infinite-dimensional variational inequalities. An exposition of infinite-dimensional variational inequalities and references can be found in Kinderlehrer and Stampacchia[55].

Dafermos[56] proved that Smith's[57] formulation of the transportation network equilibrium problem satisfied a finite-dimensional variational inequality problem. This connection allowed for the development of more general models and rigorous computational techniques for transportation network equilibrium problems, spatial price equilibrium problems, oligopolistic market equilibrium problems, as well as economic and financial equilibrium problems[5,40,58] and the references therein.

The theory of variational inequalities is especially suitable for the formulation of a wide range of supernetwork problems, since in such problems there may be several or numerous decision-makers that interact. Hence, one wishes to model and analyze not only individual behavior but the complexity of interactions among a spectrum of decision-makers, be they travelers, consumers, financial decision-makers, supply chain network decision-makers, etc.

Moreover, since many mathematical problems can be formulated as variational inequality problems, this formulation is particularly convenient since it allows for a unified treatment of equilibrium problems (governed by distinct equilibrium concepts, including game theoretic ones, as in the case of Nash[59-60] equilibria) as well as optimization problems.

In this Section, we assume that the vectors are column vectors, except where noted. We begin with a fundamental definition.

Definition: The Variational Inequality Problem

The finite-dimensional variational inequality problem, VI $(F, \mathcal{K})$, is to determine a vector $\boldsymbol{X}^* \in \mathcal{K} \subset R^n$, such that

$$\langle \boldsymbol{F}(\boldsymbol{X}^*)^{\mathrm{T}}, \boldsymbol{X} - \boldsymbol{X}^* \rangle \geqslant 0, \quad \forall \boldsymbol{X} \in \mathcal{K}$$

where F is a given continuous function from $\mathcal{K}$ to R^n, $\mathcal{K}$ is a given closed convex set, and $\langle \cdot, \cdot \rangle$ denotes the inner product in R^n, where R^n in the n-dimensional Euclidean space.

A geometric interpretation of a variational inequality problem is given in Fig. 1.

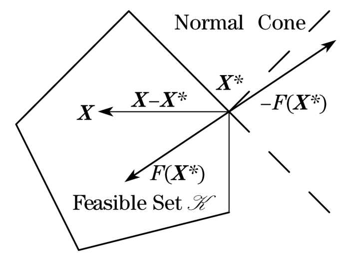

Fig. 1 Geometric Interpretation of VI($\boldsymbol{F}$, $\mathcal{K}$)

Note that, in the case of network and, hence, supernetwork problems, the underlying feasible set $\mathcal{K}$ will have a network structure.

We now discuss optimization problems, both unconstrained and constrained, and their relationships with the variational inequality problem. Proofs of the below theoretical results for variational inequalities may be found in books by Kinderlehrer and Stampacchia[55] and Nagurney[5]. For algorithms for the computation of variational inequalities, see the books by Bertsekas and Tsitsiklis[61], Nagurney[5], Patriksson[62], Nagurney and Zhang[7]. The solution of specific supernetwork models may be found in the books by Nagurney and Dong[8], Nagurney[58], and Nagurney and Qiang[9].

Relationships to Optimization Problems

Optimization problems consider explicitly an objective function to be minimized (or maximized), subject to constraints that may consist of both equalities and inequalities. Let f be a continuously differentiable function where $f: \mathcal{K} \mapsto R$. Mathematically, the statement of an *optimization problem* is

$$\text{Minimize} \quad f(\boldsymbol{X})$$
$$\text{subject to:} \ \boldsymbol{X} \in \mathcal{K}$$

The relationship between an optimization problem and a variational inequality problem is now given.

Proposition

Let $\boldsymbol{X}^*$ be a solution to the optimization problem

$$\text{Minimize} \quad f(\boldsymbol{X})$$
$$\text{subject to:} \quad \boldsymbol{X} \in \mathcal{K}$$

where f is continuously differentiable and $\mathcal{K}$ is closed and convex. Then $\boldsymbol{X}^*$ is a solution of the variational inequality problem:

$$\langle \boldsymbol{\nabla} f(\boldsymbol{X}^*)^{\mathrm{T}}, \boldsymbol{X} - \boldsymbol{X}^* \rangle \geqslant 0, \quad \forall \boldsymbol{X} \in \mathcal{K}$$

Furthermore, we have the following.

Proposition

If $f(\boldsymbol{X})$ is a convex function and $\boldsymbol{X}^*$ is a solution to VI$(\nabla f, \mathcal{K})$, then $\boldsymbol{X}^*$ is a solution to the above optimization problem.

If the feasible set $\mathcal{K} = R^n$, then the unconstrained optimization problem is also a variational inequality problem.

On the other hand, in the case where a certain symmetry condition holds, the variational inequality problem can be reformulated as an optimization problem. In other words, in the case that the variational inequality formulation of the governing (network) equilibrium conditions underlying a specific problem is characterized by a function with a symmetric Jacobian, then the solution of the equilibrium conditions and the solution of a particular optimization problem are one and the same. We first introduce the following definition and then note this relationship in a theorem.

Definition

A matrix $\boldsymbol{M}(\boldsymbol{X})$, whose elements $m_{ij}(\boldsymbol{X})$; $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, n$, are functions defined on the set $S \subset R^n$, is said to be positive semidefinite on S if

$$\boldsymbol{v}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{X})\boldsymbol{v} \geqslant 0, \quad \forall \boldsymbol{v} \in R^n, \boldsymbol{X} \in S$$

It is said to be positive definite on S if

$$\boldsymbol{v}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{X})\boldsymbol{v} > 0, \quad \forall \boldsymbol{v} \neq 0, \boldsymbol{v} \in R^n, \boldsymbol{X} \in S$$

It is said to be strongly positive definite on S if

$$\boldsymbol{v}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{X})\boldsymbol{v} \geqslant \alpha \|\boldsymbol{v}\|^2, \quad \text{for some } \alpha > 0, \quad \forall \boldsymbol{v} \in R^n, \boldsymbol{X} \in S$$

Note that if $\gamma(\boldsymbol{X})$ is the smallest eigenvalue, which is necessarily real, of the symmetric part of $\boldsymbol{M}(\boldsymbol{X})$, that is, $\frac{1}{2}[\boldsymbol{M}(\boldsymbol{X}) + \boldsymbol{M}(\boldsymbol{X})^{\mathrm{T}}]$, then it follows that (i). $\boldsymbol{M}(\boldsymbol{X})$ is positive semidefinite on S if and only if $\gamma(\boldsymbol{X}) \geqslant 0$, for all $\boldsymbol{X} \in S$; (ii). $\boldsymbol{M}(\boldsymbol{X})$ is positive definite on S if and only if $\gamma(\boldsymbol{X}) > 0$, for all $\boldsymbol{X} \in S$; and (iii). $\boldsymbol{M}(\boldsymbol{X})$ is strongly positive definite on S if and only if $\gamma(\boldsymbol{X}) \geqslant \alpha > 0$, for all $\boldsymbol{X} \in S$.

Theorem

Assume that $F(\boldsymbol{X})$ is continuously differentiable on $\mathcal{K}$ and that the Jacobian matrix

$$\nabla F(\boldsymbol{X}) = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial X_1} & \cdots & \frac{\partial F_n}{\partial X_n} \end{bmatrix}$$

is symmetric and positive semidefinite. Then there is a real-valued convex function $f: \mathcal{K} \mapsto R^1$ satisfying

$$\nabla f(\boldsymbol{X}) = F(\boldsymbol{X})$$

with $\boldsymbol{X}^*$ the solution of VI$(F, \mathcal{K})$ also being the solution of the mathematical programming problem

$$\text{Minimize} \quad f(\boldsymbol{X})$$
$$\text{subject to:} \quad \boldsymbol{X} \in \mathcal{K}$$

Hence, although the variational inequality problem encompasses the optimization problem, a variational inequality problem can be reformulated as a convex optimization problem, only when the symmetry condition and the positive semidefiniteness condition hold.

The variational inequality is the more general problem in that it can also handle a function $F(\boldsymbol{X})$ with an asymmetric Jacobian. Historically, many equilibrium problems were reformulated as optimization problems, under precisely such a symmetry assumption, including, originally, spatial price equilibrium problems as well as transportation network equilibrium problems and certain game theoretic problems. The assumption, however, in terms of applications was restrictive and precluded the more realistic modeling of multiple commodities, multiple modes and/or classes in competition. In addition, the objective function that resulted was sometimes artificial, without a clear economic interpretation, and simply a mathematical device. because of variational inequality theory, we no longer need to impose restrictive symmetry assumptions and, hence, can capture asymmetric interactions of decision-makers on complex networks.

Variational inequality theory is also a powerful tool for the qualitative analysis of solutions, notably, for obtaining existence and uniqueness results. For stability and sensitivity analysis of variational inequalities, including applications, see Nagurney[5], and the references therein.

Existence of a solution to a variational inequality problem follows from continuity of the function F entering the variational inequality, provided that the feasible set $\mathcal{K}$ is compact, as stated in the following theorem.

Theorem

If $\mathcal{K}$ is a compact (closed and bounded) convex set and $F(\boldsymbol{X})$ is continuous on $\mathcal{K}$, then the variational inequality problem admits at least one solution $\boldsymbol{X}^*$.

In the case of an unbounded feasible set $\mathcal{K}$, this Theorem is no longer applicable; the existence of a solution to a variational inequality problem can, nevertheless, be established under the subsequent condition.

Let $B_R(0)$ denote a closed ball with radius R centered at 0 and let $\mathcal{K}_R = \mathcal{K} \cap B_R(0)$. $\mathcal{K}_R$ is then bounded. By VI_R is denoted then the variational inequality problem:

Determine $\boldsymbol{X}_R^* \in \mathcal{K}_R$, such that

$$\langle F(\boldsymbol{X}_R^*)^{\mathrm{T}}, \boldsymbol{y} - \boldsymbol{X}_R^* \rangle \geqslant 0, \quad \forall\, \boldsymbol{y} \in \mathcal{K}_R$$

We now state

Theorem

$\mathrm{VI}(F, \mathcal{K})$ admits a solution if and only if there exists an $R>0$ and a solution of VI_R, $\boldsymbol{X}_R^*$, such that $\|\boldsymbol{X}_R^*\| < R$.

Although $\|\boldsymbol{X}_R^*\| < R$ may be difficult to check, one may be able to identify an appropriate R based on the particular application.

Existence of a solution to a variational inequality problem may also be established under a coercivity condition.

Qualitative properties of existence and uniqueness become easily obtainable under certain monotonicity conditions. First we outline the definitions and then present the results.

Definition: Monotonicity

$F(\boldsymbol{X})$ is monotone on $\mathcal{K}$ if

$$\langle (F(\boldsymbol{X}^1) - F(\boldsymbol{X}^2))^{\mathrm{T}}, \boldsymbol{X}^1 - \boldsymbol{X}^2 \rangle \geqslant 0, \quad \forall\, \boldsymbol{X}^1, \boldsymbol{X}^2 \in \mathcal{K}$$

Definition: Strict Monotonicity

$F(\boldsymbol{X})$ is strictly monotone on $\mathcal{K}$ if

$$\langle (F(\boldsymbol{X}^1) - F(\boldsymbol{X}^2))^{\mathrm{T}}, \boldsymbol{X}^1 - \boldsymbol{X}^2 \rangle > 0 \quad \forall\, \boldsymbol{X}^1, \boldsymbol{X}^2 \in \mathcal{K}, \quad \boldsymbol{X}^1 \neq \boldsymbol{X}^2$$

Definition: Strong Monotonicity

$F(\boldsymbol{X})$ is strongly monotone if for some $\alpha > 0$

$$\langle (F(\boldsymbol{X}^1) - F(\boldsymbol{X}^2))^{\mathrm{T}}, \boldsymbol{X}^1 - \boldsymbol{X}^2 \rangle \geqslant \alpha \|\boldsymbol{X}^1 - \boldsymbol{X}^2\|^2, \quad \forall\, \boldsymbol{X}^1, \boldsymbol{X}^2 \in \mathcal{K}$$

Definition: Lipschitz Continuous

$F(\boldsymbol{X})$ is Lipschitz continous if there exists an $L>0$, such that

$$\|F(\boldsymbol{X}^1) - F(\boldsymbol{X}^2)\| \leqslant L \|\boldsymbol{X}^1 - \boldsymbol{X}^2\|, \quad \forall\, \boldsymbol{X}^1, \boldsymbol{X}^2 \in \mathcal{K}$$

A uniqueness result is presented in the subsequent theorem.

Theorem

Suppose that $F(\boldsymbol{X})$ is strictly monotone on $\mathcal{K}$. Then the solution is unique, if one exists.

Monotonicity is closely related to positive definiteness.

Theorem

Suppose that $F(\boldsymbol{X})$ is continuously differentiable on $\mathcal{K}$ and the Jacobian matrix

$$\nabla F(\boldsymbol{X}) = \begin{bmatrix} \dfrac{\partial F_1}{\partial X_1} & \cdots & \dfrac{\partial F_1}{\partial X_n} \\ \vdots & & \vdots \\ \dfrac{\partial F_n}{\partial X_1} & \cdots & \dfrac{\partial F_n}{\partial X_n} \end{bmatrix}$$

which need not be symmetric, is positive semidefinite (positive definite). Then $F(\boldsymbol{X})$ is monotone (strictly monotone).

Proposition

Assume that $F(\boldsymbol{X})$ is continuously differentiable on $\mathcal{K}$ and that $\nabla F(\boldsymbol{X})$ is strongly positive definite. Then $F(\boldsymbol{X})$ is strongly monotone.

The following theorem provides a condition under which both existence and uniqueness of the solution to the variational inequality problem are guaranteed. Here no assumption on the compactness of the feasible set $\mathcal{K}$ is made.

Theorem

Assume that $F(\boldsymbol{X})$ is strongly monotone. Then there exists precisely one solution $\boldsymbol{X}^*$ to $\mathrm{VI}(F, \mathcal{K})$.

Hence, in the case of an unbounded feasible set $\mathcal{K}$, strong monotonicity of the function F guarantees both existence and uniqueness. If $\mathcal{K}$ is compact, then existence is guaranteed if F is continuous, and only the strict monotonicity condition needs to hold for uniqueness to be guaranteed.

In the case of supernetwork problems in which the demands are fixed, one would have a compact feasible set and, therefore, existence would be guaranteed under the sole assumption that the function F that enters the corresponding variational inequality would be continuous. Uniqueness of a solution then, in turn, would be guaranteed if the F was strictly monotone. On the other hand, as would arise in the case of elastic demand problems, in which, for example, the demand price function or the demand functions are known, the feasible set may no longer be compact and, hence, strong monotonicity of F would guarantee both existence of a solution as well as uniqueness. Of course, other conditions (as noted above) may also be relevant and guarantee existence of a solution.

Another important definition is given below.

Definition: Norm Projection

Let $\mathcal{K}$ be a closed convex set in R^N. Then for each $\boldsymbol{X} \in R^N$, there is a unique point $\boldsymbol{y} \in \mathcal{K}$, such that:

$$\|\boldsymbol{X} - \boldsymbol{y}\| \leqslant \|\boldsymbol{X} - \boldsymbol{z}\|, \quad \forall \boldsymbol{z} \in \mathcal{K}$$

and $\boldsymbol{y}$ is known as the orthogonal projection of $\boldsymbol{X}$ on the set $\mathcal{K}$ with respect to the Euclidean norm, that is:

$$\boldsymbol{y} = P_{\mathcal{K}}\boldsymbol{X} = \arg\min_{z \in \mathcal{K}} \|\boldsymbol{X} - \boldsymbol{z}\|$$

Specifically, one of the most effective algorithms for the solution of variational inequality problems is the projection method as well as the modified projection method.

We now provide an illustration of how variational inequality theory was utilized to establish that multitiered supply chain network equilibrium problems could be transformed into transportation network equilibrium problems through a supernetwork construction. Specifically, Nagurney[63] proved that supply chain network equilibrium problems (cf. Fig. 2) originated by Nagurney, Dong, and Zhang[43], and consisting of m manufacturers, n retailers, and o consumers at demand markets, each with his own explicit behavior, could be refomulated as transportation network equilibrium problems on a supernetwork, as depicted in Fig. 3. The latter consists of a single origin node 0 and as many demand nodes as there are demand markets in the supply chain. The representative functions are then mapped onto the links of the supernetwork and the conservation of flow equations are satisfied as well.

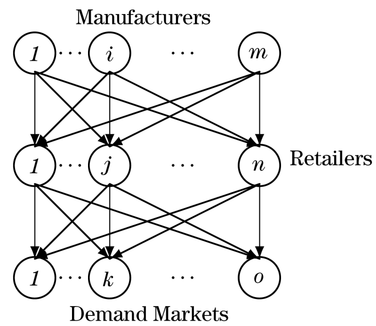

Fig. 2 The Network Structure of the Supply Chain at Equilibrium

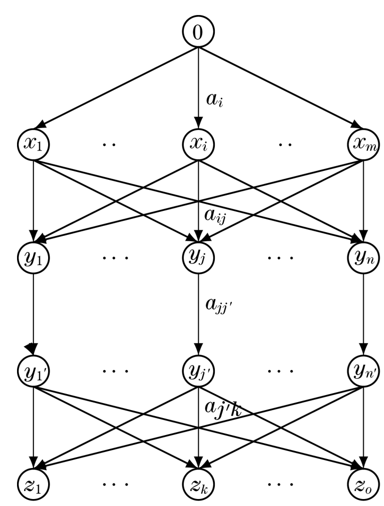

Fig. 3 The G_S Supernetwork Representation of Supply Chain Network Equilibrium

The formulation of supply chain network equilibrium problems as supernetworks provides an alternative interpretation of the governing equilibrium conditions and also enables the application of numerous algorithms that have been developed for transportation network equilibrium problems[5,62].

The world is, nevertheless, dynamic, and some of the most challenging problems that we are faced with are dynamic in nature. Ideed, we are in an era of complex dynamical network systems or complex dynamic supernetworks. Finite-dimensional variational inequality theory by itself, provides no framework for the study of the dynamics of competitive systems. Rather, it captures the system at its equilibrium state and, hence, the focus of this tool is static in nature.

Dupuis and Nagurney[64] established that, given a variational inequality problem, there is a naturally associated dynamical system, the stationary points of which correspond precisely to the solutions of the variational inequality problem. This association was first noted by Dupuis and Ishii[65]. This system, named a *projected dynamical system* by Zhang and Nagurney[6], is non-classical, in that its right-hand side, which is a projection operator, is discontinuous. The discontinuities arise because of the constraints underlying the variational inequality problem modeling the application in question. Therefore, classical dynamical systems theory is no longer applicable[66].

Here we recall some results in *projected dynamical systems* theory[7]. Projected dynamical systems theory, however, goes further than finite-dimensional variational inequality theory in that it extends the static study of equilibrium states by introducing an additional time dimension in order to allow for the analysis of disequilibrium behavior that precedes the equilibrium. Moreover, it has been used, to-date, to capture the underlying dynamics os supernetworks from complex supply chain networks to financial networks as well as the integration of social networks with financial networks as well as supply chain networks[67-69]. New models and applications are regularly being developed using this methodology.

In particular, we associate with a given variational inequality problem a projected dynamical system. The projected dynamical system is appropriate both as a dynamical model for the system whose equilibrium behavior is described by the variational inequality, and, also, because its set of stationary points coincides with the set of solutions to a variational inequality problem. In this framework, the feasibility constraints in the variational inequality problem correspond to discontinuities in the right-hand side of the differential equation, which is a projection operator.

Below we provide some important theoretical results. For additional qualitative results, in particular, stability analysis results, see Nagurney and Zhang[7]. For a discussion of the general iterative scheme and proof of convergence, see Dupuis and Nagurney[64]. For applications to dynamic spatial price equilibrium problems, oligopolistic market equilibrium problems, and transportation network equilibrium problems, see Zhang and Nagurney[6] and Nagurney and Zhang[7], and the references therein. For a focus on network economics, see Nagurney[41]. For extensions of these results to infinite-dimensional projected dynamical systems and evolutionary variational inequalities, see Cojocaru, Daniele, and Nagurney[70-71] and the books by Daniele[72] and Nagurney[58].

2.2 The Projected Dynamical System

Finite-dimensional variational inequality theory provides no framework for studying the underlying dynamics of systems, since it considers only equilibrium solutions in its formulation. Hence, in a sense, it provides a static representation of a system at its "steady state." One would, therefore, like a theoretical framework that allows for the study of a system not only at its equilibrium point, but also in a dynamical setting.

The definition of a projected dynamical system (PDS) is given with respect to a closed convex set $\mathcal{K}$, which is usually the constraint set underlying a particular application, such as, for example, supernetwork equilibrium problems, and a vector field F whose domain contains $\mathcal{K}$. Such projected dynamical systems provide mathematically convenient approximations to more "realistic" dynamical models that might be used to describe non-static behavior. The relationship between a projected dynamical system and its associated variational inequality problem with the same constraint set is then highlighted. We also, for self-containment, recall the fundamental properties of existence and uniqueness of the solution to the ordinary differential equation (ODE) that defines such a projected dynamical system.

Let $\mathcal{K}\subset R^n$ be closed and convex. Denote the boundary and interior of $\mathcal{K}$, respectively, by $\partial\mathcal{K}$ and $\mathcal{K}^0$. Given $\boldsymbol{X}\in\partial\mathcal{K}$, define the set of inward normals to $\mathscr{K}$ at $\boldsymbol{X}$ by

$$N(\boldsymbol{X}) = \{\boldsymbol{\gamma} : \|\boldsymbol{\gamma}\| = 1, \text{ and}, \langle \boldsymbol{\gamma}^{\mathrm{T}}, \boldsymbol{X} - \boldsymbol{y}\rangle \leqslant 0, \ \forall \boldsymbol{y} \in \mathscr{K}\}$$

We define $N(\boldsymbol{X})$ to be $\{\boldsymbol{\gamma} : \|\boldsymbol{\gamma}\| = 1\}$ for $\boldsymbol{X}$ in the interior of $\mathscr{K}$.

When $\mathscr{K}$ is a convex polyhedron (for example, when $\mathscr{K}$ consists of linear constraints), $\mathscr{K}$ takes the form $\bigcap_{i=1}^{Z}\mathscr{K}_i$, where each $\mathscr{K}_i$ is a closed half-space with inward normal N_i. Let $P_{\mathscr{K}}$ be the norm projection. Then $P_{\mathscr{K}}$ projects onto $\mathscr{K}$ "along N," in that if $\boldsymbol{y}\in\mathscr{K}$, then $P(\boldsymbol{y}) = \boldsymbol{y}$, and if $\boldsymbol{y}\notin\mathscr{K}$, then $P(\boldsymbol{y})\in\partial\mathscr{K}$, and $P(\boldsymbol{y}) - \boldsymbol{y} = \alpha\boldsymbol{\gamma}$ for some $\alpha > 0$ and $\boldsymbol{\gamma}\in N(P(\boldsymbol{y}))$.

Definition

Given $\boldsymbol{X}\in\mathscr{K}$ and $\boldsymbol{v}\in R^n$, define the projection of the vector $\boldsymbol{v}$ at $\boldsymbol{X}$ (with respect to $\mathscr{K}$) by

$$\boldsymbol{\Pi}_{\mathscr{K}}(\boldsymbol{X}, \boldsymbol{v}) = \lim_{\delta\to 0}\frac{(P_{\mathscr{K}}(\boldsymbol{X} + \delta\boldsymbol{v}) - \boldsymbol{X})}{\delta}$$

The class of ordinary differential equations that are of interest here take the following form

$$\dot{\boldsymbol{X}} = \boldsymbol{\Pi}_{\mathscr{K}}(\boldsymbol{X}, -F(\boldsymbol{X}))$$

where $\mathscr{K}$ is a closed convex set, corresponding to the constraint set in a particular application, and $F(\boldsymbol{X})$ is a vector field defined on $\mathscr{K}$.

A classical dynamical system, in contrast, is of the form

$$\dot{\boldsymbol{X}} = -F(\boldsymbol{X})$$

We have the following results[64]:

(i) If $\boldsymbol{X}\in\mathscr{K}^0$, then

$$\boldsymbol{\Pi}_{\mathscr{K}}(\boldsymbol{X}, -F(\boldsymbol{X})) = -F(\boldsymbol{X})$$

(ii) If $\boldsymbol{X}\in\partial\mathscr{K}$, then

$$\boldsymbol{\Pi}_{\mathscr{K}}(\boldsymbol{X}, -F(\boldsymbol{X})) = -F(\boldsymbol{X}) + \beta(\boldsymbol{X})N^*(X)$$

where

$$N^*(\boldsymbol{X}) = \arg\max_{N\in N(\boldsymbol{X})}\langle(-F(\boldsymbol{X}))^{\mathrm{T}}, -N\rangle$$

and

$$\beta(\boldsymbol{X}) = \max\{0, \langle(-F(\boldsymbol{X}))^{\mathrm{T}}, -N^*(\boldsymbol{X})\rangle\}$$

Since the right-hand side of the ordinary differential equation is associated with a projection operator, it is discontinuous on the boundary of $\mathscr{K}$. Therefore, one needs to explicitly state what is meant by a solution to an ODE with a discontinuous right-hand side.

Definition

We say that the function $\boldsymbol{X} : [0,\infty)\mapsto\mathscr{K}$ is a solution to the equation $\dot{\boldsymbol{X}} = \boldsymbol{\Pi}_{\mathscr{K}}(\boldsymbol{X}, -F(\boldsymbol{X}))$ if $\boldsymbol{X}(\cdot)$ is absolutely continuous and $\dot{\boldsymbol{X}}(t) = \boldsymbol{\Pi}_{\mathscr{K}}(\boldsymbol{X}(t), -F(\boldsymbol{X}(t)))$, save on a set of Lebesgue measure zero.

In order to distinguish between the pertinent ODEs from the classical ODEs with continuous right-hand sides, we refer to the above as ODE$(F,\mathscr{K})$.

Definition: An Initial Value Problem

For any $X_0\in\mathscr{K}$ as an initial value, we associate with ODE$(F,\mathscr{K})$ an initial value problem, IVP$(F, \mathscr{K}, X_0)$, defined as:

$$\dot{\boldsymbol{X}} = \boldsymbol{\Pi}_{\mathscr{K}}(\boldsymbol{X}, -F(\boldsymbol{X})), \quad X(0) = X_0$$

Note that if there is a solution $\phi_{X_0}(t)$ to the initial value problem IVP$(F,\mathscr{K},X_0)$, with $\phi_{X_0}(0) = X_0\in\mathscr{K}$, then $\phi_{X_0}(t)$ always stays in the constraint set $\mathscr{K}$ for $t\geqslant 0$.

We now present the definition of a projected dynamical system, governed by such an ODE$(F, \mathscr{K})$, which, correspondingly, will be denoted by PDS$(F,\mathscr{K})$.

Definition: The Projected Dynamical System

Define the projected dynamical system PDS$(F,\mathscr{K})$ as the map $\Phi: \mathscr{K}\times R\mapsto\mathscr{K}$ where

$$\Phi(\boldsymbol{X}, t) = \phi_{\boldsymbol{X}}(t)$$

solves the IVP$(F,\mathscr{K},\boldsymbol{X})$, that is,

$$\dot{\phi}_{\boldsymbol{X}}(t) = \boldsymbol{\Pi}_{\mathscr{K}}(\phi_{\boldsymbol{X}}(t), -F(\phi_{\boldsymbol{X}}(t))), \quad \phi_{\boldsymbol{X}}(0) = \boldsymbol{X}$$

The behavior of the dynamical system is now described. Please refer to Fig. 4 for an illustration of this behavior. If $\boldsymbol{X}(t)\in\mathscr{K}^0$, then the evolution of the solution is directly given in terms of F : $\dot{\boldsymbol{X}} = -F(\boldsymbol{X})$. However, if the vector field $-F$ drives $\boldsymbol{X}$ to $\partial\mathscr{K}$ (that is, for some t one has $\boldsymbol{X}(t)\in\partial\mathscr{K}$ and $-F(\boldsymbol{X}(t))$ points "out" of $\mathscr{K}$) the right-hand side of the ODE becomes the projection of $-F$ onto $\partial\mathscr{K}$. The solution to the ODE then evolves along a "section" of $\partial\mathscr{K}$, e.g., $\partial\mathscr{K}_i$ for some i. At a later time the solution may re-enter $\mathscr{K}^0$, or it may enter a lower dimensional part of $\partial\mathscr{K}$, e.g., $\partial\mathscr{K}_i\cap\partial\mathscr{K}_j$. Depending on the particular vector field F, it may then evolve within the set $\partial\mathscr{K}_i\cap\partial\mathscr{K}_j$, re-enter $\partial\mathscr{K}_i$, enter $\partial\mathscr{K}_j$, etc.

We now define a stationary or an equilibrium point. For further details, see Nagurney and Zhang[7].

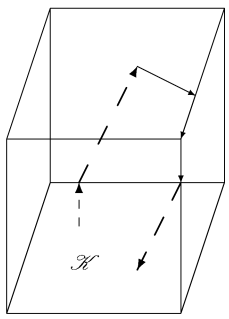

Fig.4 A Trajectory of a PDS that Evolves both on the Interior and on the Boundary of the Constraint Set $\mathcal{K}$

Definition: A Stationary Point or an Equilibrium Point

The vector $\boldsymbol{X}^* \in \mathcal{K}$ is a stationary point or an equilibrium point of the projected dynamical system PDS$(F,\mathcal{K})$ if

$$\mathbf{0} = \boldsymbol{\Pi}_{\mathcal{K}}(\boldsymbol{X}^*, -F(\boldsymbol{X}^*))$$

In other words, we say that $\boldsymbol{X}^*$ is a stationary point or an equilibrium point if, once the projected dynamical system is at $\boldsymbol{X}^*$, it will remain at $\boldsymbol{X}^*$ for all future times.

From the definition it is clear that $\boldsymbol{X}^*$ is an equilibrium point of the projected dynamical system PDS$(F,\mathcal{K})$ if the vector field F vanishes at $\boldsymbol{X}^*$. The contrary, however, is only true when $\boldsymbol{X}^*$ is an interior point of the constraint set $\mathcal{K}$. Indeed, when $\boldsymbol{X}^*$ lies on the boundary of $\mathcal{K}$, we may have $F(\boldsymbol{X}^*)\neq 0$.

Note that for classical dynamical systems, the necessary and sufficient condition for an equilibrium point is that the vector field vanish at that point, that is, that $\mathbf{0} = -F(\boldsymbol{X}^*)$.

The following theorem, due to Dupuis and Nagurney[64], states a basic connection between the static world of finite-dimensional variational inequality problems and the dynamic world of projected dynamical systems.

Theorem

Assume that $\mathcal{K}$ is a convex polyhedron. Then the equilibrium points of the PDS $(F,\mathcal{K})$ coincide with the solutions of VI$(F,\mathcal{K})$. Hence, for $\boldsymbol{X}^* \in \mathcal{K}$ and satisfying

$$\mathbf{0} = \boldsymbol{\Pi}_{\mathcal{K}}(\boldsymbol{X}^*, -F(\boldsymbol{X}^*))$$

also satisfies

$$\langle F(\boldsymbol{X}^*)^{\mathrm{T}}, \boldsymbol{X} - \boldsymbol{X}^* \rangle \geqslant 0, \quad \forall \boldsymbol{X} \in \mathcal{K}$$

This Theorem establishes the equivalence between the set of equilibria of a projected dynamical system and the set of solutions of a variational inequality problem. Moreover, it provides a natural underlying dynamics (out of equilibrium) of such systems.

Before recalling the fundamental theorem of projected dynamical systems, we introduce the following assumption needed for the theorem.

Assumption: Linear Growth Condition: There exists a $B < \infty$ such that the vector field $-F: R^n \mapsto R^n$ satisfies the linear growth condition: $\| F(\boldsymbol{X}) \| B(1 + \|\boldsymbol{X}\|)$ for $\boldsymbol{X} \in \mathcal{K}$, and also

$$\langle (-F(\boldsymbol{X}) + F(\boldsymbol{y}))^{\mathrm{T}}, \boldsymbol{X} - \boldsymbol{y} \rangle \leqslant B \|\boldsymbol{X} - \boldsymbol{y}\|^2, \quad \forall \boldsymbol{X}, \boldsymbol{y} \in \mathcal{K}$$

Theorem: Existence, Uniqueness, and Continuous Dependence Assume that the linear growth condition holds. Then

(i) For any $X_0 \in \mathcal{K}$, there exists a unique solution $X_0(t)$ to the initial value problem;

(ii) If $X_k \to X_0$ as $k \to \infty$, then $X_k(t)$ converges to $X_0(t)$ uniformly on every compact set of $[0,\infty)$.

The second statement of this Theorem is sometimes called the *continuous dependence* of the solution path to ODE$(F,\mathcal{K})$ on the initial value. By virtue of the Theorem, the PDS$(F,\mathcal{K})$ is well-defined and inhabits $\mathcal{K}$ whenever the Assumption holds.

Lipschitz continuity is a condition that plays an important role in the study of variational inequality problems. It also is a critical concept in the classical study of dynamical systems.

Lipschitz continuity implies the Assumption and is, therefore, a sufficient condition for the fundamental properties of projected dynamical systems stated in the Theorem.

Cruz, Nagurney, and Wakolbinger[68] developed a projected dynamical systems model of a supernetwork consisting of a social network and a global supply chain network (see Fig. 5). They modeled the dynamic evolution of the product flows between tiers of the supply chain network as well as that of the relationship levels. They applied the Euler method, which is induced by the general iterative scheme of Dupuis and Nagurney[64], in order to track the dynamic trajectories of both the product flows and the relationship levels in discrete time.

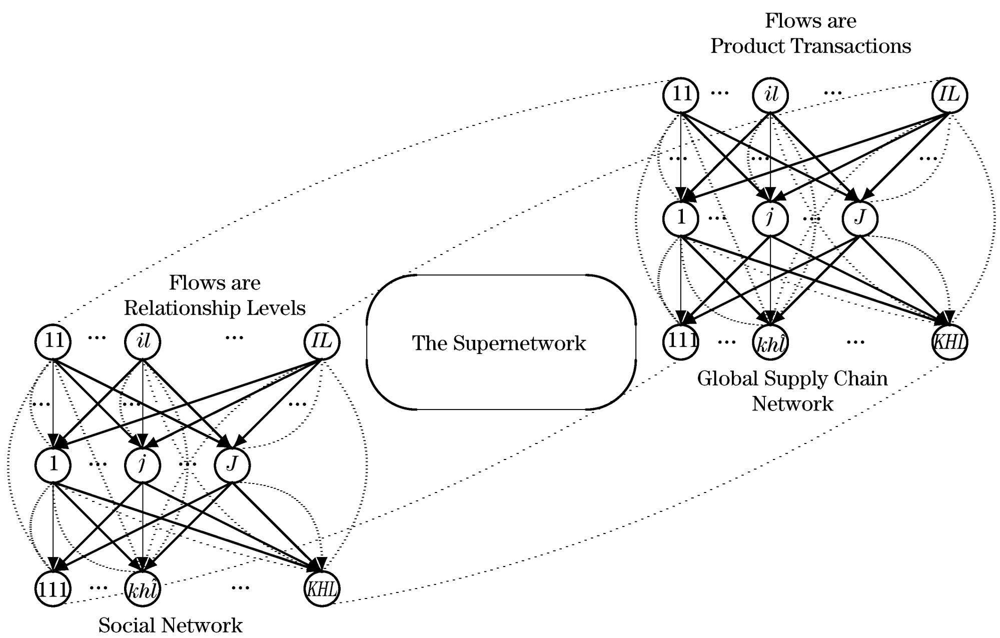

Fig.5　The Multilevel Supernetwork Structure of the Integrated Global Supply Chain Network/Social Network System

3　Supernetwork Theory Yields Answers to Questions Raised Years Ago

A theory is only as powerful as its ability to explain, to illuminate, and to answer questions.

In 1952, Copeland raised the question of how does money flow and wondered whether it flows like water or electricity.

In 2007, Liu and Nagurney established that multitiered financial networks, as introduced by Nagurney and Ke[45], and depicted in Fig. 6, could be transformed into transportation networks over appropriately constructed supernetworks, as displayed in Fig. 7.

In 1956, Beckmann, McGuire, and Winsten hypothesized in their classical book, *Studies in the Economics of Transportation*, that electric power generation and distribution networks could be modeled as transportation networks. In 2005, Nagurney and Liu[73] proved, using supernetworks, that it was, indeed, possible, yielding new insights into the behavior of complex networks as well as the commonality of their structure. Please refer to Fig. 8 and 9.

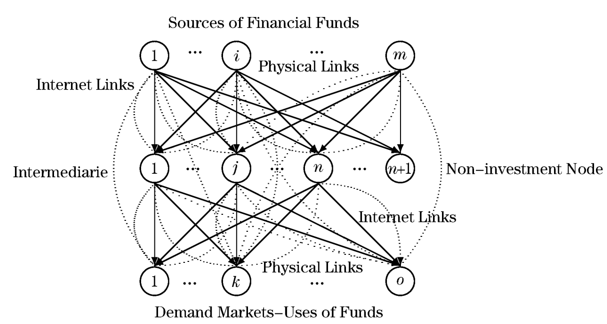

Fig.6　The Structure of the Financial Network with Inter-mediation and with Electronic Transactions

These theoretical results enabled Liu and Nagurney[74] to model and solve an integrated electric power supply chain and fuel market network for New England, using variational inequalities. They used real data for 82 power generators, who own and operate 573 power plants, with 5 different fuels, and 10 regions, and utilized hourly demand price data for the entire month of July 2006. The computational results well-reproduced the true price data, showing both the theoretical and empirical validity of the modeling and computational approach.

Hence, through the use of supernetworks, we were able to show that both money, as well as electricity, flow like transportation flows, answering, thus, questions posed more than a half a century earlier! The full model descriptions and proofs can be found in the above cited papers.

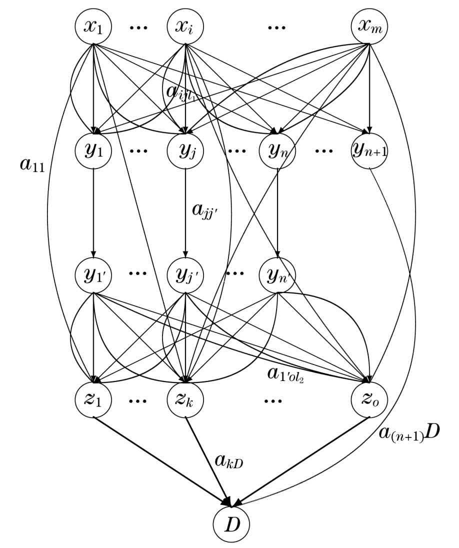

Fig.7 The G_S Supernetwork Representation of Financial Network Equilibrium

By developing supernetwork representations, we enable not only alternative interpretation of the governing equilibrium conditions, but also can avail ourselves of a plethora of algorithms that have been developed for the solution of transportation network equilibrium problems. In addition, such connections have yielded not only new, dynamic models but also allowed for large-scale empirical modeling and applications.

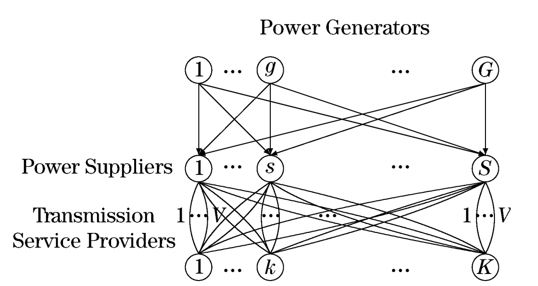

Fig.8 The Electric Power Supply Chain Network

In Nagurney et al.[73], the supernetwork model for electric power supply chains was then utilized to develop an evolutionary, that is, time-dependent (and infinite-dimensional) variational inequality model to demonstrate the evolution of electric power flows as the demand varied. It is important to note that evolutionary variational inequalities were also utilized to develop a dynamic model of the Internet by Nagurney, Parkes, and Daniele[18] and to formulate the time-dependent Braess paradox[75].

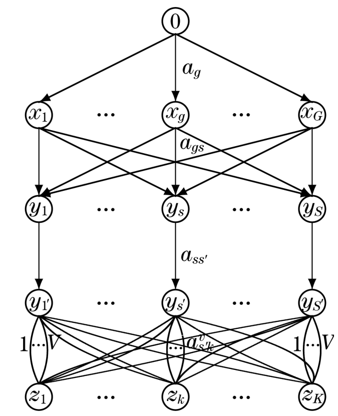

Fig.9 The $\mathcal{G}_S$ Supernetwork Representation of Electric Power Supply Chain Network Equilibrium

4 Network Efficiency/Performance Assessment and Vulnerability Analysis

The topic of complex networks, as a body of research and practice, has attracted researchers and practitioners from different disciplines from operations research/management science, computer science, and engineering, sociology, and more recently, from physics and even biology[8,15,76]. This is due not only to the wide applications but also, in part, to a spectrum of catastrophic events such as 9/11, the North American electric power blackout in 2003, Hurricane Katrina in 2005, the Minneapolis bridge collapse in 2007, the Mediterranean cable disruption in 2008, Cyclone Nargis and the Sichuan Earthquake in China in 2008, the H1N1 pandemic in 2009, the earthquakes in Haiti and Chile in 2010, and the triple earthquake/tsunami/nuclear disaster in Japan in 2011, among others, all of which have drawn great attention to network vulnerability and fragility[9,77-78].

As noted in the Introduction, the number of disasters is increasing globally, as well as the number of people affected by disasters, posing new challenges for emergency and disaster preparedness and for the critical network infrastructure itself from transportation to telecommunications. For example, between 2000 and 2004 the average annual number of disasters was 55% higher than in the period 1994 through 1999, with 33% more humans affected in the former period than in the latter[9,79]. The International Strategy for Disaster Reduction (2006) determined that approximately 150 million people required assistance, because of disasters, in 2005, with 157 million requiring assistance in 2006.

Nagurney and Qiang[9,78] noted that the recent theories of scale-free and small-world networks in complex network research have enhanced our understanding of some of the behavior and the vulnerability of specific real-world networks[80-82]. However, most network vulnerability studies have concentrated on the topological characteristics of networks, such as the connectivity or the shortest path length of the network[83], and the references therein. Although the topological structure of a network provides information regarding network vulnerability, the flow on a network is also an important indicator, as are the operational and economic aspects, such as the flow-induced costs, and the behavior of users both prior and post any disruptions. Barabási[84] emphasized that: "To achieve that [understanding of complexity] we must move beyond structure and topology and start focusing on the dynamics that take place along the links. Networks are only skeletons of complexity, the highways for various processes that make our world hum."

Latora and Marchiori[85-87] proposed a network efficiency measure that exhibited advantages over several existing network measures and applied their measure to study the (MBTA) Boston subway network and the Internet. Their measure, however, considers only geodesic information and does not capture information contained in network flows, the associated costs, and users' behavior, be it according to centralized or decentralized decision-making principles.

Clearly, complex networks are not simply "graphs" in which nodes are connected by links (whether directed or undirected, as is relatively common in social network analysis). It is not just the roads, telecommunication links, transmission lines, etc., that matter but also the behavior of the decision-makers, coupled with the induced flows, the incurred costs (and prices or disutilities) as they cooperate or compete over time.

Supernetwork theory allows us to apply consistent network efficiency performance measures to identify the importance and ranking of network components, whether nodes, links, or combinations thereof. Such metrics are critically important, in that one can assess which network components should be prioritized for maintenance (or reconstruction and recovery) for both emergency preparedness and disaster recovery purposes.

Nagurney and Qiang[51-53,88] introduced a consistent network efficiency/performance measure and importance indicator to incorporate such important network characteristics as decision-making induced flows and costs in order to quantify the importance of network components in network systems, ranging from congested urban transportation networks to electric power supply chains and financial networks. This network measure has significant advantages and captures the reality of networks today in that it captures congestion. In addition, the measure can handle both fixed and elastic demand network problems[88-89] plus time-dependent, dynamic networks[18,90]. The topic of *centrality* of nodes (and that of links or "edges") in a network is a major issue in network characterization[91] with contributors from sociology[92-97]. The behavior of decision-makers, and the readjustment after nodal or link removal, is not captured in their centrality measures, but it is in ours.

4.1 A Unified Network Performance Measure

Before we recall the unified network performance measure of Qiang and Nagurney[89] (see also Nagurney and Qiang[51-53,88,90,98]) we review an important property that such a measure should have.

Network Performance Property:

The performance/efficiency measure for a given network should be nonincreasing with respect to the equilibrium disutility for each O/D pair, holding the equilibrium disutilities for the other O/D pairs constant.

This performance measure is for network operations under decentralized decision-making behavior, such as congested urban transportation networks, the Internet, certain supply chains and financial networks, as well as electric power generation and distribution networks. Hence, the overarching modeling paradigm, and the fundamental network model is that of the transportation network equilibrium model (with either fixed or elastic demands)[56,99] with the underlying behavioral principle being that of user-optimization (as opposed to system-optimization). The full description of the methodology can be found in Nagurney and Qiang[9,78] and the references therein (where we also address robustness and provide measures based on system-optimizing behavior). Here we note some of the highlights from an extensive body of research.

Given this property of a network performance measure, the unified network performance measure is as below.

Definition: A Unified Network Performance Measure

The network performance/efficiency measure, $\varepsilon(\mathcal{G}, \boldsymbol{d})$, for a given network topology $\mathcal{G}$ and the equilibrium (or fixed) demand vector $\boldsymbol{d}$, is:

$$\varepsilon = \varepsilon(\mathcal{G}, \boldsymbol{d}) = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W}$$

where recall that n_W is the number of O/D pairs in the network, and d_w and λ_w denote, for simplicity, the equilibrium (or fixed) demand and the equilibrium disutility for O/D pair w, respectively.

In applying ε, it is important to note that the elimination of a link is treated by removing that link from the network while the removal of a node is managed by removing the links entering and exiting that node. If the removal results in no path connecting an O/D pair, we just assign the demand for that O/D pair (either fixed or elastic) to an abstract path at a cost of infinity.

As established in Qiang and Nagurney[89], under certain assumptions, the unified measure collapses to the Latora and Marchiori[85] measure, which, however, considers neither explicit demands nor flows and is as follows:

Definition: The Latora and Marchiori Measure

Let n be the number of nodes in $\mathcal{G}$. Then the Latora and Marchiori network efficiency measure E is

$$E = E(\mathcal{G}) = \frac{1}{n(n-1)} \sum_{i \neq j \in \mathcal{G}} \frac{1}{d_{ij}}$$

where d_{ij} is the shortest path length (geodesic distance) between nodes i and j.

Theorem

If positive demands exist for all pairs of nodes in the network $\mathcal{G}$, and each of these demands is equal to 1 and if d_{ij} is set equal to λ_w, where $w = (i, j)$, for all $w \in W$ then the proposed network efficiency measure ε and the E measure are one and the same.

The proof of the above theorem assumes that d_{ij} is equal to the corresponding λ_w, which is not unreasonable. The ε measure, however, is more general since it captures the flows on networks and their reallocation, in the case of disruptions, through the demands, disutilities, and costs.

As discussed in Nagurney and Qiang[9], for a network with fixed demands, it is easy to verify that the unified measure ε is well-defined. In a network with elastic demands, when there is a disconnected O/D pair w, we have, from the above discussion, that the associated "path cost" of the abstract path, say, r, $C_r(x^*)$, is equal to infinity. If the disutility functions are known, according to the equilibrium conditions, we then have that $C_r(x^*) > \lambda_w(d^*)$, and, hence, $x_r^* = 0$, so that $d_w^* = 0$, which leads to the conclusion of $d_w^* / \lambda_w = 0$. Therefore, the disconnected O/D pair w makes zero "contribution" to the efficiency measure and ε is well-defined in both the fixed and elastic demand cases. We can expect a network to get disconnected in the case of disasters and, consequently, our measure has the essential feature that it is well-defined even in such situations.

The unified measure ε has the following interpretation in the case of transportation networks. The equilibrium O/D pair disutility, λ_w, is proportional to the (travel) time between each O/D pair w. d_w is the equilibrium demand (in terms of total vehicles) between each O/D pair w. Therefore, d_w/λ_w is the (vehicle) throughput between O/D pair w. $\varepsilon(\mathcal{G},\boldsymbol{d})$ is the average (vehicle) throughput on the network $\mathcal{G}$ with demand vector $\boldsymbol{d}$. The higher the throughput that a network has, the better its performance and the more efficient it is. For general networks, ε is actually the average demand to price ratio. When $\mathcal{G}$ and $\boldsymbol{d}$ are fixed, a network is more efficient if it can satisfy a higher demand at a lower price.

4.2 The Importance of Network Components

With thenetwork performance/efficiency measure, we can quantifiably determine the importance of network components by studying their impact on the network efficiency through their removal. The network efficiency can be expected to deteriorate when a critical network component is eliminated from the network. We expect that the removal of a critical network component will cause greater impact than that of a trivial one. The importance of a network component is defined, following Qiang and Nagurney[89], as follows.

Definition: Importance of a Network Component

The importance of a network component $g\in\mathcal{G}$, $I(g)$, is measured by the relative network efficiency drop after g is removed from the network:

$$I(g)=\frac{\Delta\varepsilon}{\varepsilon}=\frac{\varepsilon(\mathcal{G},\boldsymbol{d})-\varepsilon(\mathcal{G}-g,\boldsymbol{d})}{\varepsilon(\mathcal{G},\boldsymbol{d})}$$

where $\mathcal{G}-g$ is the resulting network after component g is removed from network $\mathcal{G}$.

The upper bound of the importance of a network component is 1. The higher the value, the more important a network component is.

It is important to also recognize that the Nagurney and Qiang measure has been applied in practice. Indeed, as noted in Nagurney and Qiang[78], it has been applied by Schulz[100] to evaluate highways in Germany and found to outperform several existing measures. The above importance indicator can also be used to evaluate additions to a network in terms of the improvement of a network's efficiency/performance. Walsh[101] applied ε to determine how efficient the proposed North Dublin metro would be.

Also, this measure has been adapted to evaluate the performance of supply chains as well as financial networks[98]. In addition, it has generated extensions to assess supply chain networks under risk and uncertainty[9,78], and the references therein.

This research has now turned to network design issues with the full recognition that different applications may require different performance measures to capture the relevant objectives. For example, supply chain network design that captures both capacity expansion (or link construction) as well as operational issues was initiated by Nagurney[102] who used a system-optimization for network design and redesign. Extensions have included the inclusion of competition in an oligopolistic framework by Nagurney[103] as well as the inclusion of uncertainty and risk by Nagurney, Yu, and Qiang[104] as in the case of critical needs products that would be vital in the case of emergencies and disasters. In addition, multiproduct supply chain network design models have been developed by Nagurney, Yu, and Qiang[77] with a focus on healthcare. It is important to note that the behavior of users must be captured in network design since as first demonstrated by Braess[16], the addition of a new road to a network, under travelers' use-optimizing behavior, may make all travelers worse off in terms of travel cost/time!

Network design may also be accomplished through the merging of networks, as can occur in the case of mergers and acquisitions[105]. In such settings multicriteria decision-making may play an important role and should be part of the synergy measures, as done by Nagurney and Woolley[106] who included environmental issues to assess synergy (in addition to cost) in the context of mergers and acquisitions. More recently, Liu and Nagurney[107] showed how risk can also be quantified, in a supernetwork framework, to assess a priori the possible synergy of mergers and acquisitions.

The supernetwork approach has also been applied to assess the potential teaming of organizations for humanitarian logistics by Nagurney and Qiang[9]. A bi-criteria metric to assess supply chain network performance for critical needs products under capacity and demand disruptions was recently developed by Qiang and Nagurney[108].

Recent novel applications have also been studied in the fashion and apparel industries where multicriteria decision-making with especial emphasis on time issues as well as competition is very relevant[109-110].

5 Some Other Successes and Suggestions for Future Research

Methodologies that break down boundaries between disciplines are especially valued in science. Some notable recent successes have included the use of networks and variational inequality theory to model complex food webs, using what the authors, Mullon, Shin, and Cury[111], deemed to be a network economics approach of Nagurney[5]. In such ecological predator-prey networks, which may have a bipartite structure, may even express cannibalism, depending on the species, or be quite complex and large-scale, as in the case of ocean fisheries, the flows are biomass flows and the individual species are represented as nodes. Directed links then capture the prey-predator relationships. Nagurney and Nagurney[112] were able to demonstrate that the bipartite predator-prey networks actually correspond to "classical" spatial price equilibrium problems, well-known in economics and regional science. Nagurney and Nagurney[113] then developed a projected dynamical systems model of multitiered predator-prey networks and proved, using supernetworks, that they are identical to multitiered supply chain network problems (see also Fig. 2) both in equilibrium (and disequilibrium). Hence, ecological predator-prey networks are truly nature's supply chains. This research helps to bridge the disciplines of economics and ecology as well as operations research/management science.

Fascinatingly, projected dynamical systems theory is now being utilized in neuroscience to model the brain, one of the most complex networks in existence, and is also being applied to robotics[114].

Finally, we note that projected dynamical systems theory of Dupuis and Nagurney[63] and Nagurney and Zhang[7] and the references therein has now become a powerful methodology in economics for evolutionary game theory and evolutionary dynamics[115-116].

It is the era of supernetworks, complex dynamical systems, and risk. Research in the future will break down additional barriers between disciplines and, it is expected, will unveil new frontiers in risk modeling and dynamic network systems. Clearly, supernetwork theory will also advance and will play an important role in helping to explain the complexity and the interconnectedness of the world around us.

Acknowledgments

The author thanks the Editors of this journal for the opportunity to write this invited paper.

She also thanks all of her doctoral students, both present and former, and her collaborators, for the honor and privilege of working with them.

Finally, she acknowledges support from funding agencies, including the National Science Foundation, the Rockefeller Foundation, the AT & T Foundation, and the Fulbright program, as well as the John F. Smith Memorial Fund at the Isenberg School of Management at the University of Massachusetts Amherst, which provided support over the years that made the research highlighted in this paper possible.

References:

[1] BAR-GERA H. Origin-based algorithm for the traffic assignment problem [J]. Transportation Science, 2002, 36: 398-417.

[2] WU J H, FLORIAN M, HE S G. EMME/2 Implementation of the SCAG-Ⅱ Model: Data Structure, System Analysis and Computation, INRO Solutions Internal Report[R]. Montreal: the Southern California Association of Governments, 2000.

[3] Miniwatts Marketing Group[EB/OL]. [2011-04-

06]. http://www.internetworldstats.com/stats.htm extracted.
[4] SCHRANK D, LOMAX T, TURNER S. TTI's urban mobility report[R]. Texas: Texas Transportation Institute, The Texas A & M University System, 2010.
[5] NAGURNEY A. Network Economics: A Variational Inequality Approach[M]. Second and revised edition. Boston: Kluwer Academic Publishers, 1999.
[6] ZHANG D, NAGURNEY A. On the stability of projected dynamical systems[J]. Journal of Optimization Theory and its Applications, 1995, 85: 97-124.
[7] NAGURNEY A, ZHANG D. Projected Dynamical Systems and Variational Inequalities with Applications[M]. Boston: Kluwer Academic Publishers, 1996.
[8] NAGURNEY A, DONG J. Supernetworks: Decision-Making for the Information Age[M]. Cheltenham: Edward Elgar Publishers, 2002.
[9] NAGURNEY A, QIANG Q. Fragile Networks: Identifying Vulnerabilities and Synergies in an Uncertain World[M]. Hoboken: John Wiley & Sons, 2009.
[10] NAGURNEY A. Navigating the Network Economy [M]//OR/MS Today. Atlanta: Lionheart Publishing Company, 2000: 74-75.
[11] NAGURNEY A. Distinguished Faculty Lecture: Networks for fun and profit. University of Massachusetts, Amherst[EB/OL]. [2000-04-05]. http://intra.som.umass.edu/fomgt/dislec.pdf.
[12] DAFERMOS S. The traffic assignment problem for multimodal networks [J]. Transportation Science, 1972, 6: 73-87.
[13] BECKMANN M, McGUIRE C B, WINSTEN C B. Studies in the Economics of Transportation[M]. New Haven: Yale University Press, 1956.
[14] BOYCE D E, MAHMASSANI H S, NAGURNEY A. A retrospective on Beckmann, McGuire, and Winsten's studies in the economics of transportation[J]. Papers in Regional Science, 2005, 84: 85-103.
[15] ROUGHGARDEN T. The Price of Anarchy[M]. Cambridge: MIT Press, 2005.
[16] BRAESS D. Uber ein paradoxon der verkehrsplanung [J]. Unternehmenforschung, 1968, 12: 258-268.
[17] BRAESS D, NAGURNEY A, WAKOLBINGER T. On a paradox of traffic planning, translation of the original D. Braess paper from German to English[J]. Transportation Science, 2005, 39: 446-450.
[18] NAGURNEY A, PARKES D, DANIELE P. The Internet, evolutionary variational inequalities, and the time-dependent Braess paradox [J]. Computational Management Science, 2007, 4: 355-375.
[19] DAFERMOS S. Integrated equilibrium flow models for transportation planning[M]//Florian M A. Traffic Equilibrium Methods, Lecture Notes in Economics and Mathematical Systems, 118. New York: Springer-Verlag, 1976: 106-118.
[20] SHEFFI Y. Transportation Network Equilibrium with Discrete Choice Models[D]. Cambridge: Civil Engineering Department, Massachusetts Institute of Technology, 1978.
[21] SHEFFI Y, DAGANZO C F. Hypernetworks and supply-demand equilibrium obtained with disaggregate demand models[J]. Transportation Research Record, 1978, 673: 113-121.
[22] SHFFI Y, DAGANZO C F. Computation of equilibrium over transportation networks: The case of disaggregate demand models[J]. Transportation Science, 1980, 14: 155-173.
[23] SHEFFI Y. Urban Transportation Networks[M]. Englewood Cliffs, New Jersy: Prentice-Hall, 1985.
[24] WARDROP J G. Some theoretical aspects of road traffic research[C]//Proceedings of the Institute of Civil Engineers, Part Ⅱ, 1952: 325-278.
[25] DENNING P J. The science of computing: Supernetworks[J]. American Scientist, 1985, 73: 127-129.
[26] SCHUBERT L, GOEBEL R, CERCONE N. The structure and organization of a semantic net for xomprehension and inference[M]//FINDLER M V. Associative Networks: Representation and Use of Knowledge by Computers. New York: Academic Press, 1979.
[27] FALLOWS J. The java theory, The Atlantic Online, The Atlantic Monthly/Digital Edition [EB/OL]. [2011-03-10]. http://www2.theatlantic.com/issues/96mar/java/java.htm.
[28] QUESNAY F. Tableau Economique[M]. 1758. reproduced in facsimile with an introduction by H. Higgs by the British Economic Society, 1895.
[29] COURNOT A A. Researches into the Mathematical Principles of the Theory of Wealth[M]. 1838. English translation, London: MacMillan, 1897.
[30] PIGOU A C. The Economics of Welfare[M]. London: MacMillan, 1920.
[31] DAFERMOS S, NAGURNEY A. Isomorphism between spatial price and traffic equilibrium models[R]. Providence, Rhode Island: Lefschetz Center for Dynamical Systems, Brown University, 1985.
[32] EUKE S. Equilibrium among spatially separated markets: solution by electronic analogue[J]. Econometri-

ca,1951,10:40－47.

[33] SAMUELSON P A. Spatial price equilibrium and linear programming[J]. American Economic Review, 1952,42:283－303.

[34] TAKAYAMA T, JUDGE G G. An intertemporal price equilibrium model[J]. Journal of Farm Economics, 1964,46:477－484.

[35] TAKAYAMA T, JUDGE G G. Spatial and Temporal Price and Allocation Models[M]. Amsterdam: North-Holland,1971.

[36] FLORIAN M, LOS M. A new look at static spatial price equilibrium models[J]. Regional Science and Urban Economics,1982,12:579－597.

[37] DAFERMOS S, NAGURNEY A. Sensitivity analysis for the general spatial economics equilibrium problem [J]. Operations Research,1984,32:1069－1086.

[38] DAFERMOS S, NAGURNEY A. Oligopolistic and competitive behavior of spatially separated markets[J]. Regional Science and Urban Economics, 1987, 17: 245－254.

[39] NAGURNEY A, SIOKOS S. Financial Networks: Statics and Dynamics [M]. Heidelberg: Springer-Verlag,1997.

[40] NAGURNEY A E. Innovations in Financial and Economic Networks [M]. Cheltenham: Edward Elgar Publishing,2003.

[41] NAGURNEY A. Networks in Finance[M]//Belseley D, Kontoghiorghes E. Handbook of Computational Econometrics. Cichester: John Wiley & Sons, 2009: 429－486.

[42] NAGURNEY A, LOO J, DONG J, et al. Supply chain networks and electronic commerce: a theoretical perspective[J]. Netnomics,2002,4:187－220.

[43] NAGURNEY A, DONG J, ZHANG D. A supply chain network equilibrium model[J]. Transportation Research E,2002,38:281－303.

[44] NAGURNEY A, KE K, CRUZ J, et al. Dynamics of supply chains: A multilevel (logistical/informational/financial) network perspective[J]. Environment & Planning B,2002,29:795－818.

[45] NAGURNEY A, KE K. Financial networks with electronic transactions: Modeling, analysis, and computations[J]. Quantitative Finance,2003,3:71－87.

[46] NAGURNEY A, TOYASAKI F. Supply chain supernetworks and environmental criteria[J]. Transportation Research D,2003,8:185－213.

[47] NAGURNEY A, DONG J, MOKHTARIAN P L. Multicriteria network equilibrium modeling with variable weights for decision-making in the Information Age with applications to telecommuting and teleshopping [J]. Journal of Economic Dynamics and Control, 2002,26:1629－1650.

[48] NAGURNEY A, CRUZ J, MATSYPURA D. Dynamics of global supply chain supernetworks[J]. Mathematical and Computer Modelling,2003,37:963－983.

[49] NAGURNEY A, CRUZ J. International financial networks with intermediation: Modeling, analysis, and computations[J]. Computational Management Science,2003,1:31－58.

[50] NOVEEN A, HARTENSTEIN V, CHUONG C M. Gene networks and supernetworks: Evolutionarily conserved gene interactions[M]//CHUONG C M. Molecular Basis of Epithelial Appendage Morphogenesis. Austin: Landes Bioscience,1998:371－391.

[51] NAGURNEY A, QIANG Q. A network efficiency measure for congested networks[J]. Europhysics Letters,2007,79,38005:1－5.

[52] NAGURNEY A, QIANG Q. A transportation network efficiency measure that captures flows, behavior, and costs with applications to network component importance identification and vulnerability[C]//Proceedings of the 18th Annual POMS Conference, Dallas, Texas,2007.

[53] NAGURNEY A, QIANG Q. Robustness of transportation networks subject to degradable links[J]. Europhysics Letters,2007,80,68001:1－6.

[54] HARTMAN P, STAMPACCHIA G. On some nonlinear elliptic differential functional equations[J]. Acta Mathematica,1966,115:271－310.

[55] KINDERLEHER D, STAMPACCHIA G. An Introduction to Variational Inequalities and Their Applications [M]. New York: Academic Press,1980.

[56] DAFERMOS S. Traffic equilibrium and variational inequalities[J]. Transportation Science,1980,14:42－54.

[57] SMITH M J. Existence, uniqueness, and stability of traffic equilibria[J]. Transportation Research, 1979, 13B:295－304.

[58] NAGURNEY A. Supply Chain Network Economics: Dynamics of Prices, Flows, and Profits[M]. Cheltenham, England: Edward Elgar Publishing,2006.

[59] NASH J F. Equilibrium points in n-person games[J]. Proceedings of the National Academy of Sciences USA,1950,36:48－49.

[60] NASH J F. Noncooperative games[J]. Annals of Mathematics,1951,54:286－298.

[61] BERTSEKAS D P, TSITSIKLIS J N. Parallel and Dis-

tributed Computation[M]. Englewood Cliffs：Prentice-Hall，1989.

[62] PATRIKSSON M. The Traffic Assignment Problem [M]. Utrecht，The Netherlands：VSP，1994.

[63] NAGURNEY A. On the relationship between supply chain and transportation network equilibria：A supernetwork equivalence with computations[J]. Transportation Research E，2006，42：293－316.

[64] DUPUIS P，NAGURNEY A. Dynamical systems and variational inequalities[J]. Annals of Operations Research，1993，44：9－42.

[65] DUPUIS P，ISHII H. On Lipschitz continuity of the solution mapping to the Skorokhod Problem，with applications[J]. Stochastics and Stochastic Reports，1991，35：31－62.

[66] HARTMAN P. Ordinary Differential Equations[M]. New York：John Wiley & Sons，1964.

[67] WAKOLBINGER T，NAGURNEY A. Dynamic supernetworks for the integration of social networks and supply chains with electronic commerce：Modeling and analysis of buyer-seller relationships with computations[J]. Netnomics，2004，6：153－185.

[68] CRUZ J M，NAGURNEY A，WAKOLBINGER T. Financial engineering of the integration of global supply chain networks and social networks with risk management[J]. Naval Research Logistics，2006，26：674－696.

[69] NAGURNEY A，WAKOLBINGER T，ZHAO L. The evolution and emergence of integrated social and financial networks with electronic transactions：A dynamic supernetwork theory for the modeling，analysis，and computation of financial flows and relationship levels[J]. Computational Economics，2006，27：353－393.

[70] COJOCARU M G，DANIELE P，NAGURNEY A. Projected dynamical systems and evolutionary variatinal inequalities via Hilbert spaces with applications[J]. Journal of Optimization Theory and Applications，2005，27：1－13.

[71] COJOCARU M G，DANIELE P，NAGURNEY A. Double-layered dynamics：A unified theory of projected dynamical systems and evolutionary variational inequalities[J]. European Journal of Operational Research，2006，175：494－507.

[72] DANIELE P. Dynamic Networks and Variational Inequalities [M]. Cheltenham：Edward Elgar Publishing，2006.

[73] NAGURNEY A，LIU Z，Cojocaru M G，et al. Dynamic electric power supply chains and transportation networks：An evolutionary variational inequality formulation[J]. Transportation Research E，2007，43：624－646.

[74] LIU Z，NAGURNEY A. An integrated electric power supply chain and fuel market network framework：theoretical modeling with empirical analysis for New England[J]. Naval Research Logistics，2009，56：600－624.

[75] PAS E，PRINCIPIO S I. Braess' paradox：Some new insights[J]. Transportation Research B，1997，31：265－276.

[76] NEWMAN M E J，BARABÁSI A L，WSTTS D J. The Structure and Dynamics of Networks[M]. Princeton：Princeton University Press，2006.

[77] NAGURNEY A，YU M，QIANG Q. Multiproduct supply chain network design with applications to healthcare [R]. Amherst：Isenberg School of Management，University of Massachusetts，2009.

[78] NAGURNEY A，QIANG Q. Fragile networks：Identifying vulnerabilities and synergies in an uncertain age [J]. International Transactions in Operational Research，2011，in press.

[79] BALCIK B，BEAMON B. Facility location in humanitarian relief[J]. International Journal of Logistics：Research and Applications，2008，11：101－121.

[80] AMARAL L A N，SCALA A，Barthèlémy M，et al. Classes of small-world networks[J]. Proceedings of the National Academy of Sciences USA，2000，97：11149－11152.

[81] CHASSIN D P，POSSE C. Evaluating North American electric grid reliability using the Barabàsi-Albert network model[J]. Physica A，2005，355：667－677.

[82] HOLMGREN A J. A framework for vulnerability assessment of electric power systems[M]//Murray A，Grubesic T. Reliability and Vulnerability in Critical Infrastructure：A Quantitative Geographic Perspective. New York：Springer，2007：31－55.

[83] CALLAWAY D S，NEWMAN M E J，STROGATZ S H，et al. Network robustness and fragility：Presentation on random graphs[J]. Physical Review Letters，2000，85：546－571.

[84] BARABÁSI A L. Linked：How Everything is Connected to Everything Else and What It Means[M]. New York：Plume，2003.

[85] LATORA V，MARCHIORI M. Efficient behavior of small-world networks[J]. Physics Review Letters，2001，87，198701：1－4.

[86] LATORA V, MARCHIORI M. Is the Boston subway a small-world network? [J]. Physica A, 2002, 314: 109-113.

[87] LATORA V, MARCHIORI M. How the science of complex networks can help developing strategies against terrorism[J]. Chaos, Solitons and Fractals, 2004, 20: 69-75.

[88] NAGURNEY A, QIANG Q. A network efficiency measure with application to critical infrastructure networks[J]. Journal of Global Optimization, 2008, 40: 261-275.

[89] QIANG Q, NAGURNEY A. A unified network performance measure with importance identification and the ranking of network components[J]. Optimization Letters, 2008, 2: 127-142.

[90] NAGURNEY A, QIANG Q. An efficiency measure for dynamic networks with application to the Internet and vulnerability analysis[J]. Netnomics, 2008, 9: 1-20.

[91] BARRAT A, BARTHÉLÈMY M, PASTOR-SATORRAS R, et al. The architecture of complex weighted networks[J]. Proceedings of the National Academy of Sciences USA, 2004, 101: 3747-3752.

[92] FREEMAN L C. Centrality in social networks: Conceptual clarification[J]. Social Networks, 1979, 1: 215-239.

[93] BONACICH P. Factoring and weighting approaches to status scores and clique identification[J]. Journal of Mathematical Sociology, 1972, 2: 13-20.

[94] FREEMAN L C, BORGATTI S P, WHITE D R. Centrality in valued graphs: A measure of betweenness based on network flow[J]. Social Networks, 1991, 13: 141-154.

[95] NEWMAN M E J. Analysis of weighted networks[J]. Physical Review E, 2004, 70, 056131.

[96] BARRAT A, BARTHÉLÈMY M, VESPIGNANI A. The effects of spatial constraints on the evolution of weighted complex networks[J]. Journal of Statistical Mechanics: Theory and Experiment, 2005, P05003.

[97] DALL'ASTA L, BARRAT A, BARTHÉLEMY M, et al. Vulnerability of weighted networks[J]. Journal of Statistical Mechanics, 2006, P04006.

[98] NAGURNEY A, QIANG Q. Identification of critical nodes and links in financial networks with intermediation and electronic transactions[M]//KONTOGHIORGHES E J, RUSTEM B, WINKER P. Computational Methods in Financial Engineering. Berlin: Springer, 2008, 273-297.

[99] DAFERMOS S. The general multimodal traffic equilibrium problem with elastic demand[J]. Networks, 1982, 12: 57-72.

[100] SCHULZ C. Identification of critical transportation infrastructures, Forum DKKV/CEDM, Disaster Reduction in Climate Change[R]. Germany: Karlsruhe University, 2007.

[101] WALSH J R. Route optimization: how efficient will the proposed north Dublin metro be? [J]. ERCIM News, 2009, 79: 45-46.

[102] NAGURNEY A. Optimal supply chain network design and redesign at minimal total cost and with demand satisfaction[J]. International Journal of Production Economics, 2010, 128: 200-208.

[103] NAGURNEY A. Supply chain network design under profit maximization and oligopolistic competition[J]. Transportation Research E, 2010, 46: 281-294.

[104] NAGURNEY A, YU M, QIANG Q. Supply chain network design for critical needs with outsourcing[J]. Regional Science, 2011, 90: 123-142.

[105] NAGURNEY A. A system-optimization perspective for supply chain network integration: The horizontal merger case[J]. Transportation Research E, 2009, 45: 1-15.

[106] NAGURNEY A, WOOLLEY T. Environmental and cost synergy in supply chain network integration in mergers and acquisitions, in Sustainable Energy and Transportation Systems[C]//EHRGOTT M, NAUJOKS B, STEWART T J, et al. Proceedings of the 19th International Conference on Multiple Criteria Decision Making, Lecture Notes in Economics and Mathematical Systems. Berlin: Springer, 2010: 51-78.

[107] LIU Z, NAGURNEY A. Risk reduction and cost synergy in mergers and acquisitions via supply chain network integration[J]. Journal of Financial Decision Making, 2011, in press.

[108] QIANG Q, NAGURNEY A. A bi-criteria measure to assess supply chain network performance for critical needs under capacity and demand disruptions[J]. Transportation Research A, 2011, in press.

[109] NAGURNEY A, YU M. Fashion supply chain management through cost and time minimization from a network perspective[M]//CHOI T M. Fashion Supply Chain Management: Industry and Business Analysis, Hershey: IGI Global, 2011, in press.

[110] NAGURNEY A, YU M. Sustainable fashion supply chain management under oligopolistic competition and brand differentiation[J]. International Journal of

Production Economics，2011，in press.

[111] MULLON C，SHIN Y，CURY P. NEATS：A network economic approach to trophic systems[J]. Ecological Modelling，2009，220：3033－3045.

[112] NAGURNEY A，NAGURNEY L S. Spatial price equilibrium and food webs：The economics of predator-prey networks[C]//Proceedings of the 2011 IEEE Conference on Supernetworks and System Management，2011，in press.

[113] NAGURNEY A，NAGURNEY L S. Dynamics and equilibria of ecological predator-prey networks as nature's supply chains[J]. Transportation Research E，2011，in press.

[114] GIRARD B，TABAREAU N，PHAM Q C，et al. Where neuroscience and dynamic system theory meet autonomous robotics：A contracting basal ganglia model for action selection[J]. Neural Networks，2008，21：628－641.

[115] SANDHOLM W. Excess payoff dynamics and other well-behaved evolutionary dynamics[J]. Journal of Economic Theory，2005，124：149－170.

[116] SANDHOLM W. Population Games and Evolutionary Dynamics[M]. Cambridge：MIT Press，2011.

[117] COPELAND M A. A Study of Moneyflows in the United States[R]. New York：National Bureau of Economic Research，1952.

[118] DAFERMOS S C，SPARROW F T. The traffic assignment problem for a general network[J]. Journal of Research of the National Bureau of Standards，1969，73B：91－118.

[119] Illinois State Bar Association. Advisory opinion on professional conduct，Opinion No. 96-10，Topic：Electronic Communications；Confidentiality of Client Information；Advertising and Solicitation[EB/OL]. [1997－05－16]. http：//www. chicago-law. net/cyberlaw/electric. html.

[120] International Strategy for Disaster Reduction. Press Release[EB/OL]，(2006－01－30)[2009－01－05]. http：//www. unisdr. org/eng/media-room/press-release/2006/ PR-2006-02-Disasters\-increase-18-percent-2005-but-death-rates-drop. pdf.

[121] LIU Z，NAGURNEY A. Financial networks with intermediation and transportation network equilibria：A supernetwork equivalence and reinterpretation of the equilibrium conditions with computations[J]. Computational Management Science，2007，4：243－281.

[122] NAGURNEY A，LIU Z. Transportation network equilibrium reformulations of electric power networks with computations[R]. Amherst：Isenberg School of Management，University of Massachusetts，2005.

[123] QIANG Q，NAGURNEY A，DONG J. Modeling of supply chain risk under disruptions with performance measurement and robustness analysis[M]//WU T，BLACKHURST J. Managing Supply Chain Risk and Vulnerability：Tools and Methods for Supply Chain Decision Makers. London，England：Springer，2009：91－111.

超网络：复杂性科学

Anna Nagurney
（马萨诸塞大学 伊森伯格管理学院 财务和运营管理系，马萨诸塞　01003）

摘要：概述了超网络理论，并突出强调了其在科学理论、研究方法和实际应用中的重要成果.本文旨在展示超网络理论应用于建模、分析和研究现代经济社会复杂网络的深度和广度.同时提出了对超网络理论未来研究方向的建议.

关键词：超网络；网络；复杂网络；复杂性
中图分类号：N 94　　**文献标志码：**A

作者介绍

Anna Nagurney 美国布朗大学博士，马萨诸塞大学 Amherst 分校伊森伯格管理学院财务和运营管理系约翰史密斯讲座教授、超网络虚拟中心创始人兼主任，国际区域规划科学协会院士、世界银行顾问，兼任美国哈佛大学雷德克里夫高等研究院特聘教授、意大利贝拉吉奥研究中心洛克菲勒基金会特聘讲座教授、奥地利斯布鲁克大学经济学院 Fulbright 杰出讲座教授、瑞典皇家理工学院交通与区域分析系客座教授.

在国际上率先提出超网络研究.研究重点是网络系统理论和应用，在运输和物流管理、关键基础设施、经济和金融领域都有领先研究成果.已出版多部学术专著，在国际知名期刊发表论文 130 多篇.

荣获国际运筹管理学会和美国国家科学基金会贡献奖等多项国际奖.曾获哈佛大学 Radcliffe 研究院高级研究员基金、洛克菲勒基金、Fulbright 杰出学者基金、AT&T 工业生态学研究基金、美国国家科学基金等多项重大基金.

现任《网络系统新方向》丛书、《计算经济学进展》丛书编辑，并任 Networks, Journal of Economic Dynamics and Control, Computational Economics, Annals of Regional Science, International Journal of High Performance Computing Applications, Journal of Computational Optimization in Economics and Finance, Optimization Letters, International Journal of Sustainable Transportation, International Transactions in Operational Research 等多项国际知名期刊编委.

超网络研究中心简介

美国 Anna Nagurney 教授是超网络(supernetworks)研究的首创者，她在 2002 年的论著中，把高于而又超于现存网络(“above and beyond”existing networks)的网络称为超网络.随着研究工作的开展，她在马萨诸塞大学创立了超网络研究中心(The Virtual Center for Supernetworks).该中心开展了范围广泛的研究，诸如:供应链与社会网络结合的超网络、电子商务中的供应链超网络、闭环供应链超网络、交通网络、社会网络、生物网络、计算机网络、通信网络、电力网络等等.该中心的研究已产生了广泛的国际影响，已在中国、意大利、冰岛、瑞典、加拿大、英国等国家开展了相应的研究，建立了相应的组织.

上海理工大学管理学院在 2010 年 7 月成立了超网络研究中心(中国)，该中心由上海市东方学者董琼(June Dong)教授任主任.董琼教授是 Anna Nagurney 教授 2002 年超网络论著的合作者，是超网络研究领域的著名教授.

本刊编者