文章编号:1007-6735(2016)02-0109-06 DOI:10.13255/j.cnki.jusst.2016.02.002

带势函数的双调和不等式组的整体解的不存在性

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摘要: 研究了一类带势函数的双调和不等式组的整体解的不存在性.在一定条件下,通过选择合适的测试函数,利用 Young 不等式、Hölder 不等式及 Sobolev 嵌入定理等,得到解的先验估计,并应用这一估计证明不等式组的整体解的不存在性.

关键词: 双调和; 测试函数; 整体解
中图分类号: O 175.25 **文献标志码:** A

Nonexistence of Global Solutions for a Class of Biharmonic Inequalities

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Abstract: The nonexistence of golbal solutions for a class of biharmonic inequalities was studied. Under certain conditions, by choosing appropriate test functions and using the Young's inequality, Hölder's inequality, Sobolev type inequalities, a priori estimate for the solutions was obtained and the nonexistence of the golbal solutions for the systems was proven.

Keywords: *biharmonic*; *test function*; *global solution*

1 问题的提出

研究不等式组

$$\begin{cases} -\Delta^2 u + \lambda V_1(x)uv \geqslant a(x)u^q, \\ -\Delta^2 v + \lambda V_2(x)uv \geqslant b(x)v^q, \end{cases} \quad x \in R^N \quad (1)$$

其中,$N>4$,$q>2$,$\lambda>0$,且 $a(x)$,$b(x)$,$V_1(x)$,$V_2(x)\in L^1_{\mathrm{loc}}(R^N)$是非负函数,使得存在正常数 C_1,C_2,C_3,C_4,θ,使 $0\leqslant\theta<4$,且对足够大的 $|x|$,有

$$V_1(x)\leqslant\frac{C_1}{|x|^4},\ V_2(x)\leqslant\frac{C_2}{|x|^4},$$

$$a(x)\geqslant\frac{C_3}{|x|^\theta},\ b(x)\geqslant\frac{C_4}{|x|^\theta} \quad (2)$$

假设函数 $V_1(x)$,$V_2(x)$满足 Hardy 不等式,即

$$\lambda_H\int u^2V(x)\mathrm{d}x\leqslant\int(\Delta u)^2\mathrm{d}x,$$

$$\forall\ u\in W^{2,2}(R^N) \quad (3)$$

其中,$\lambda_H=\left(\frac{N(N-4)}{4}\right)^2$.

文献[1]给出了四阶 Schrödinger 不等式

收稿日期: 2015-01-08
基金项目: 沪江基金资助项目(B14005)
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$$-\Delta^2 u+\lambda V(x)u \geqslant a(x)u^q,\ x\in R^N \quad (4)$$

的先验估计.

Brezis[2]研究了下列问题：

令 $1<p<\infty$,对每个 $f\in L^1_{\mathrm{loc}}(R^N)$,在 $D'(R^N)$ 中,存在唯一的 $u\in L^p_{\mathrm{loc}}(R^N)$满足

$$-\Delta u+|u|^{p-1}u=f(x)$$

而且,如果在 R^N 上,几乎处处有 $f\geqslant 0$,则几乎处处有 $u\geqslant 0$.

对每一个 $f\in L^1_{\mathrm{loc}}(R^N)$,则在 $D'(R^N)$上,存在唯一的 $u\in L^p_{\mathrm{loc}}(R^N)$及 $g(u)\in L^1_{\mathrm{loc}}(R^N)$满足方程

$$-\Delta u+g(u)=f(x)$$

Quittner[3]研究了下列方程：

$$\begin{cases}-\Delta u=u^r v^p, & x\in \Omega\\ -\Delta v=u^q v^s, & x\in\Omega\\ u=v=0, & x\in\partial\Omega\end{cases}$$

其中,Ω 是 R^N 上的光滑有界区域,$p,q,r,s\geqslant 0$,$pq\neq(1-r)(1-s)$,p,q,r,s 满足合适的次临界度条件.文献[3]证明了先验估计以及方程正解的存在性.

Yarur[4]研究了下列方程：

在 $\Omega\subset R^N$ 上,

$$\begin{cases}-\Delta u=a(x)v^p\\ -\Delta v=b(x)u^q\end{cases}$$

其中,$p\geqslant 1,q\geqslant 1,pq>1,a(x),b(x)\in L^\infty_{\mathrm{loc}}(\Omega)$为非负函数,文献[4]证明了方程组非负解的不存在性.

Mitidieri 等[5]通过选取合适的测试函数,证明了拟线性椭圆方程正解的不存在性.

Gidas 等[6-7]给出了刘维尔定理与各种半线性、拟线性椭圆方程的先验估计解之间的联系.有关拟线性算子的更一般的结论见文献[8-9].

受以上文献的启发,本文研究双调和不等式组(1),通过解的先验估计来证明不等式组的整体解的不存在性.给出一些辅助的引理,证明双调和不等式组的整体解的不存在性.

2 预备知识和重要引理

$R>0$.在本文中假设 $A_R=\{x\in R^N\mid R\leqslant|x|<2R\}$和 $B_R=\{x\in R^N\mid 0<|x|<R\}$.记 $\eta=(\eta_1,\eta_2,\eta_3,\cdots,\eta_N)$,其中,$\eta_1,\eta_2,\cdots,\eta_N$ 是正整数.

定义 1 令 $N>4$,称 u,v 是不等式组(1)的弱解,若它们满足下列条件：

a. $u,v\in W^{2,2}_{\mathrm{loc}}(R^N)\cap L^\infty_{\mathrm{loc}}(R^N)$；

b. $\begin{cases}-\int\Delta u\Delta\varphi+\int\lambda uvV_1(x)\varphi\geqslant\int u^q a(x)\varphi,\\ -\int\Delta v\Delta\varphi+\int\lambda vuV_2(x)\varphi\geqslant\int v^q b(x)\varphi,\end{cases}$

$\forall\varphi\geqslant 0,\varphi\in C_0^\infty(R^N)$.

引理 1[1] 令 $R>0$, $m>1$, 且 $\varphi(x)=\left[\varphi_0\left(\frac{|x|}{R}\right)\right]^m$, $x=(x_1,x_2,\cdots,x_N)\in R^N$,其中

$$\varphi_0(z)=\begin{cases}1, & 0\leqslant|z|\leqslant 1\\ 0, & |z|\geqslant 2\end{cases}\quad (5)$$

$\varphi_0\in C_0^\infty(R^N)$,且当 $0\leqslant z<2$ 时,$0\leqslant\varphi_0(z)\leqslant 1$.令 $0<|\eta|<m$,则有

$$D^\eta\varphi(x)=\sum_{s\in S}\left[\varphi_0\left(\frac{|x|}{R}\right)\right]^{m-|\eta|}C_s\left(\frac{|x|}{R}\right)\cdot \frac{1}{|x|^{\beta_s}R^{\kappa_s}}\prod_{i=1}^N x_i^{\gamma_s^i},\ \forall x\in A_R \quad (6)$$

有限集 S 依赖于 η,其中,$C_s\in C_0^\infty(R^N)$,γ_s^i,$\beta_s,\kappa_s\geqslant 0$,且是取决于 $s\in S$ 的整数.

$$\beta_s+\kappa_s-\sum_{i=1}^N\gamma_s^i=|\eta|,\ \forall s\in S$$

而且,若 $d>0$,有

$$|D^\eta\varphi(x)|^d\leqslant C\frac{1}{R^{|\eta|d}}\left[\varphi_0\left(\frac{|x|}{R}\right)\right]^{d(m-|\eta|)},\quad \forall x\in A_R \quad (7)$$

其中,C 与 R 无关.

推论 1 令 $\alpha>0$, $R_0>0$, $2<q<\frac{N-\theta+\alpha(4-\theta)}{N-4}$,$\chi=\frac{\alpha+q}{\alpha+1}$,$\chi'=\frac{\alpha+q}{q-1}$是 Höder 共轭指数,$y=q-1$,$y'=\frac{q-1}{q-2}$是 Höder 共轭指数,引入引理 1 中的 φ,m 足够大时,则有下列 3 个式子成立：

a. $\lim\limits_{R\to\infty}\int_{A_R}|x|^{\frac{\theta\chi'}{\chi}}\frac{|\nabla\varphi|^{4\chi'}}{\varphi^{3\chi'+\frac{\chi'}{\chi}}}=0$；

b. $\lim\limits_{R\to\infty}\int_{A_R}|x|^{\frac{\theta\chi'}{\chi}}\frac{(\Delta\varphi)^{2\chi'}}{\varphi^{\chi'+\frac{\chi'}{\chi}}}=0$；

c. $\lim\limits_{R\to\infty}\int_{B_R}|x|^{\frac{\theta y'}{y}+\frac{\theta\chi'}{\chi}y'-4y'\chi'}\varphi<\infty$.

证明 参考文献[1],应用引理 1 中的式(7),对 $\forall R>0$,有

a.

$$\int_{A_R}|x|^{\frac{\theta\chi'}{\chi}}\frac{|\nabla\varphi|^{4\chi'}}{\varphi^{3\chi+\frac{\chi'}{\chi}}}\leqslant CR^{\frac{\theta\chi'}{\chi}-4\chi'+N}\quad (8)$$

b.

$$\int_{A_R}|x|^{\frac{\theta\chi'}{\chi}}\frac{(\Delta\varphi)^{2\chi'}}{\varphi^{\chi'+\frac{\chi'}{\chi}}}\leqslant CR^{\frac{\theta\chi'}{\chi}-4\chi'+N}\quad (9)$$

c. 运用球坐标换元,有

$$\int_{B_R}|x|^{\frac{\theta y'}{y}+\frac{\theta\chi'}{\chi}y'-4y'\chi'}=C\int_{R_0}^R t^{\frac{\theta y'}{y}+\frac{\theta\chi'}{\chi}y'-4y'\chi'}t^{N-1}\mathrm{d}t\leqslant$$

$$\frac{C}{\frac{\theta y'}{y}+\frac{\theta\chi'}{\chi}y'-4y'\chi'+N}\cdot$$
$$\left[R^{\frac{\theta y'}{y}+\frac{\theta\chi'}{\chi}y'-4y'\chi'+N}-R_0^{\frac{\theta y'}{y}+\frac{\theta\chi'}{\chi}y'-4y'\chi'+N}\right] \tag{10}$$

其中，常数 C 不依赖于 R.

由于

$$2<q<\frac{N-\theta+\alpha(4-\theta)}{N-4}$$

所以

$$\frac{\theta\chi'}{\chi}-4\chi'+N=\frac{\theta(1+\alpha)}{q-1}-\frac{4(q+\alpha)}{q-1}+N<0$$

且

$$\frac{\theta y'}{y}+\frac{\theta\chi'}{\chi}y'-4y'\chi'+N<0$$

在式(8)～(10)中，指数为负，令 $R\to\infty$，即 a，b，c 得证.

引理 2　令 u,v 是不等式组(1)的弱解，$f,g\in W^{2,2}_{\mathrm{loc}}(R^N)$是非负函数，则有

$$\begin{cases}-\int\Delta u\Delta(f\psi)+\lambda\int uvV_1(x)f\psi\geqslant\int u^q a(x)f\psi,\\ -\int\Delta v\Delta(g\psi)+\lambda\int uvV_2(x)g\psi\geqslant\int v^q b(x)g\psi,\end{cases}$$
$$\forall\,\psi\geqslant0,\ \psi\in C_0^\infty(R^N)$$

证明　令$(\rho_n)_{n\geqslant0}$是正则化序列，定义

$$\begin{cases}f_n=(f\psi)*\rho_n\in C_0^\infty(R^N)\\ g_n=(g\psi)*\rho_n\in C_0^\infty(R^N)\end{cases}$$

且 Ω 是 R^N 中的开球，使得 $\mathrm{spt}(f_n)\cup\mathrm{spt}(\psi)\in\Omega$，$\mathrm{spt}(g_n)\cup\mathrm{spt}(\psi)\in\Omega$，$\forall\ n>0$. 因为，$f\psi|_\Omega\in W_0^{2,2}(\Omega)$，$g\psi|_\Omega\in W_0^{2,2}(\Omega)$，则存在子列$(f_n)_{n>0}$，$(g_n)_{n>0}$，使得在 $W^{2,2}(\Omega)$中有：$f_n\to f\psi$，$g_n\to g\psi$.

在 Ω 中，几乎处处有

$$f_n\to f\psi,\ g_n\to g\psi \tag{11}$$

由 Hardy 不等式(3)，可知

$$\lambda_H\int_\Omega|f_n-f_m|^2V_1(x)\leqslant\int_\Omega[\Delta(f_n-f_m)]^2$$
$$\lambda_H\int_\Omega|g_n-g_m|^2V_2(x)\leqslant\int_\Omega[\Delta(g_n-g_m)]^2$$

因此，$(f_n(V_1(x))^{\frac12})_{n\geqslant0}$，$(g_n(V_2(x))^{\frac12})_{n\geqslant0}$是 $L^2(\Omega)$上的柯西列，则有另一子列$(f_n)_{n>0}$，$(g_n)_{n>0}$，使得$\exists\,W_1,W_2\in L^2(\Omega)$，对$\forall\,n>0$，在 Ω 中，几乎处处有

$$f_n(V_1(x))^{\frac12}\leqslant W_1,\ g_n(V_2(x))^{\frac12}\leqslant W_2 \tag{12}$$

应用柯西施瓦茨不等式

$$\int_\Omega|f_n-f|V_1(x)uv\leqslant\left[\int_\Omega|f_n-f|^2V_1(x)\right]^{\frac12}\cdot\left[\int_\Omega u^2v^2V_1(x)\right]^{\frac12}$$
$$\int_\Omega|g_n-g|V_2(x)uv\leqslant\left[\int_\Omega|g_n-g|^2V_2(x)\right]^{\frac12}\cdot\left[\int_\Omega u^2v^2V_2(x)\right]^{\frac12}$$

由式(11)和式(12)及 Lebesgue 控制收敛定理，在 $L^2(R^N)$上，有

$$f_n(V_1(x))^{\frac12}\to f(V_1(x))^{\frac12}\psi$$
$$g_n(V_2(x))^{\frac12}\to g(V_2(x))^{\frac12}\psi$$

由 $u,v\in L^\infty_{\mathrm{loc}}(\Omega)$，且 $V_1(x)$，$V_2(x)\in L^1_{\mathrm{loc}}(R^N)$，有

$$\begin{cases}\int_\Omega f_nuvV_1(x)\to\int_\Omega fuv\psi V_1(x)\\ \int_\Omega g_nuvV_2(x)\to\int_\Omega guv\psi V_2(x)\end{cases} \tag{13}$$

再次应用柯西施瓦茨不等式，有

$$\left|\int_\Omega\Delta u\Delta f_n-\int_\Omega\Delta u\Delta(f\psi)\right|\leqslant\left[\int_\Omega(\Delta u)^2\right]^{\frac12}\cdot\left[\int_\Omega|\Delta(f_n-f\psi)|^2\right]^{\frac12}$$
$$\left|\int_\Omega\Delta v\Delta g_n-\int_\Omega\Delta v\Delta(g\psi)\right|\leqslant\left[\int_\Omega(\Delta v)^2\right]^{\frac12}\cdot\left[\int_\Omega|\Delta(g_n-g\psi)|^2\right]^{\frac12}$$

由 $u,v\in W^{2,2}_{\mathrm{loc}}(R^N)$及式(11)，有

$$\begin{cases}\int_\Omega\Delta u\Delta(f_n)\to\int_\Omega\Delta u\Delta(f\psi)\\ \int_\Omega\Delta v\Delta(g_n)\to\int_\Omega\Delta v\Delta(g\psi)\end{cases} \tag{14}$$

由于 $f_n,g_n\in C_0^\infty(R^N)$，由定义 1 中的 b，得

$$\begin{cases}-\int\Delta u\Delta(f_n)+\lambda\int f_nuvV_1(x)\geqslant\int f_nu^qa(x)\\ -\int\Delta v\Delta(g_n)+\lambda\int g_nuvV_2(x)\geqslant\int g_nv^qb(x)\end{cases} \tag{15}$$

因此，应用式(13)和式(14)，有

$$\begin{cases}\liminf\int f_nu^qa(x)\leqslant\lambda\int uvf\psi V_1(x)-\int\Delta u\Delta(f\psi)\\ \liminf\int g_nv^qb(x)\leqslant\lambda\int uvg\psi V_2(x)-\int\Delta v\Delta(g\psi)\end{cases}$$

再应用 Fatou 引理，引理 2 得证.

引理 3　令 u,v 是不等式组(1)的弱解，在 R^N 上，几乎处处有：$u\geqslant0$，$v\geqslant0$，$\Delta u\geqslant0$，$\Delta v\geqslant0$. 令 $\alpha>2$，且 $2<q<\frac{N+\alpha(4-\theta)-\theta}{N-4}$，则有 $u^{q+\alpha}a(x)\in L^1(R^N)$，$v^{q+\alpha}b(x)\in L^1(R^N)$.

证明　易得 $u^\alpha,v^\alpha\in W^{2,2}_{\mathrm{loc}}(R^N)$，因此，应用引

理2,则有

$$\begin{cases}\int u^{q+\alpha}a(x)\varphi \leqslant -\int \Delta u\Delta(u^{\alpha}\varphi)+\lambda\int u^{\alpha+1}vV_1(x)\varphi \\ \int v^{q+\alpha}b(x)\varphi \leqslant -\int \Delta v\Delta(v^{\alpha}\varphi)+\lambda\int v^{\alpha+1}uV_2(x)\varphi\end{cases}$$

将不等式组的右边第1项展开移项,有

$$\begin{cases}\int u^{q+\alpha}a(x)\varphi+\int \alpha u^{\alpha-1}(\Delta u)^2\varphi+\int \alpha(\alpha-1)\cdot \\ \quad u^{\alpha-2}\Delta u|\nabla u|^2\varphi \leqslant -2\alpha\int u^{\alpha-1}\Delta u\nabla u\nabla\varphi- \\ \quad \int u^{\alpha}\Delta u\Delta\varphi+\lambda\int u^{\alpha+1}vV_1(x)\varphi \\ \int v^{q+\alpha}b(x)\varphi+\int \alpha v^{\alpha-1}(\Delta v)^2\varphi+\int \alpha(\alpha-1)\cdot \\ \quad v^{\alpha-2}\Delta v|\nabla v|^2\varphi \leqslant -2\alpha\int v^{\alpha-1}\Delta v\nabla v\nabla\varphi- \\ \quad \int v^{\alpha}\Delta v\Delta\varphi+\lambda\int v^{\alpha+1}uV_2(x)\varphi\end{cases} \tag{16}$$

对不等式组(16)的右边3项分别进行估计,取 $\varepsilon_1>0$,指数$\frac{1}{2}$,对不等式组的第1项应用Young不等式,有

$$\begin{cases}2\alpha\left|\int u^{\alpha-1}\Delta u\nabla u\nabla\varphi\right| \leqslant \frac{\varepsilon_1^2}{2}\int \alpha(\alpha-1)u^{\alpha-2}\cdot \\ \quad \Delta u|\nabla u|^2\varphi+\frac{4\alpha^2}{2\varepsilon_1^2\alpha(\alpha-1)}\int u^{\alpha}\Delta u\frac{|\nabla\varphi|^2}{\varphi} \\ 2\alpha\left|\int v^{\alpha-1}\Delta v\nabla v\nabla\varphi\right| \leqslant \frac{\varepsilon_1^2}{2}\int \alpha(\alpha-1)v^{\alpha-2}\cdot \\ \quad \Delta v|\nabla v|^2\varphi+\frac{4\alpha^2}{2\varepsilon_1^2\alpha(\alpha-1)}\int v^{\alpha}\Delta v\frac{|\nabla\varphi|^2}{\varphi}\end{cases} \tag{17}$$

再对式(17)的右边第2项继续应用Young不等式,取 $\varepsilon_2>0$,指数$\frac{1}{2}$,则有

$$\begin{cases}2\alpha\left|\int u^{\alpha-1}\Delta u\nabla u\nabla\varphi\right| \leqslant \frac{\varepsilon_1^2}{2}\int \alpha(\alpha-1)u^{\alpha-2}\cdot \\ \quad \Delta u|\nabla u|^2\varphi+\frac{\varepsilon_2^2}{2}\int \alpha u^{\alpha-1}(\Delta u)^2\varphi+ \\ \quad k_1(\alpha)\int u^{\alpha+1}\frac{|\nabla\varphi|^4}{\varphi^3} \\ 2\alpha\left|\int v^{\alpha-1}\Delta v\nabla v\nabla\varphi\right| \leqslant \frac{\varepsilon_1^2}{2}\int \alpha(\alpha-1)v^{\alpha-2}\cdot \\ \quad \Delta v|\nabla v|^2\varphi+\frac{\varepsilon_2^2}{2}\int \alpha v^{\alpha-1}(\Delta v)^2\varphi+ \\ \quad k_1(\alpha)\int v^{\alpha+1}\frac{|\nabla\varphi|^4}{\varphi^3}\end{cases} \tag{18}$$

其中,$k_1(\alpha)=\left[\frac{4\alpha^2}{2\varepsilon_1^2\alpha(\alpha-1)}\right]^2\frac{1}{2\alpha\varepsilon_2^2}$.

接下来对不等式组(16)的第2项进行估计,取 $\varepsilon_3>0$,指数$\frac{1}{2}$,应用Young不等式,有

$$\begin{cases}\left|\int u^{\alpha}\Delta u\Delta\varphi\right| \leqslant \frac{\varepsilon_3^2}{2}\int \alpha u^{\alpha-1}(\Delta u)^2\varphi+ \\ \quad \frac{1}{2\varepsilon_3^2\alpha}\int u^{\alpha+1}\frac{(\Delta\varphi)^2}{\varphi} \\ \left|\int v^{\alpha}\Delta v\Delta\varphi\right| \leqslant \frac{\varepsilon_3^2}{2}\int \alpha v^{\alpha-1}(\Delta v)^2\varphi+ \\ \quad \frac{1}{2\varepsilon_3^2\alpha}\int v^{\alpha+1}\frac{(\Delta\varphi)^2}{\varphi}\end{cases} \tag{19}$$

将式(17)和式(18)代入式(16),有

$$\begin{cases}\int u^{q+\alpha}a(x)\varphi+k_3(\alpha)\int \alpha(\alpha-1)u^{\alpha-2}\Delta u\cdot \\ \quad |\nabla u|^2\varphi+k_2(\alpha)\int \alpha u^{\alpha-1}(\Delta u)^2\varphi \leqslant k_1(\alpha)\cdot \\ \quad \int u^{\alpha+1}\frac{|\nabla\varphi|^4}{\varphi^3}+\frac{1}{2\varepsilon_3^2\alpha}\int u^{\alpha+1}\frac{(\Delta\varphi)^2}{\varphi}+ \\ \quad \lambda\int u^{\alpha+1}vV_1(x)\varphi \\ \int v^{q+\alpha}b(x)\varphi+k_3(\alpha)\int \alpha(\alpha-1)v^{\alpha-2}\Delta v\cdot \\ \quad |\nabla v|^2\varphi+k_2(\alpha)\int \alpha v^{\alpha-1}(\Delta v)^2\varphi \leqslant k_1(\alpha)\cdot \\ \quad \int v^{\alpha+1}\frac{|\nabla\varphi|^4}{\varphi^3}+\frac{1}{2\varepsilon_3^2\alpha}\int v^{\alpha+1}\frac{(\Delta\varphi)^2}{\varphi}+ \\ \quad \lambda\int v^{\alpha+1}uV_2(x)\varphi\end{cases} \tag{20}$$

其中,$k_2(\alpha)=\left(1-\frac{\varepsilon_2^2}{2}-\frac{\varepsilon_3^2}{2}\right)$,$k_3(\alpha)=1-\frac{\varepsilon_1^2}{2}$.

取共轭指数 $\chi=\frac{q+\alpha}{\alpha+1}$,$\chi'=\frac{q+\alpha}{q-1}$及 $y=q-1$,$y'=\frac{q-1}{q-2}$,令 $R_0>0$,使得当 $|x|\geqslant R_0$ 时,式(2)成立,取 $\delta_1,\delta_2,\delta_3,\delta_4>0$,对不等式组(20)的右边3项应用Hölder不等式以及Young不等式,有:

a.

$$\left|k_1(\alpha)\int u^{\alpha+1}\frac{|\nabla\varphi|^4}{\varphi^3}\right| \leqslant k_1(\alpha)\left[\int \frac{u^{q+\alpha}}{|x|^{\theta}}\varphi\right]^{\frac{1}{\chi}}\cdot$$
$$\left[\int |x|^{\frac{\theta\chi'}{\chi}}\frac{|\nabla\varphi|^{4\chi'}}{\varphi^{3\chi'+\frac{\chi'}{\chi}}}\right]^{\frac{1}{\chi'}} \leqslant \frac{\delta_1^{\chi}}{C_3\chi}\int u^{q+\alpha}a(x)\varphi+$$
$$\frac{[k_1(\alpha)]^{\chi'}}{\chi'\delta_1^{\chi'}}\left[\int |x|^{\frac{\theta\chi'}{\chi}}\frac{|\nabla\varphi|^{4\chi'}}{\varphi^{3\chi'+\frac{\chi'}{\chi}}}\right]$$

$$\left|k_1(\alpha)\int v^{\alpha+1}\frac{|\nabla\varphi|^4}{\varphi^3}\right| \leqslant k_1(\alpha)\left[\int \frac{v^{q+\alpha}}{|x|^{\theta}}\varphi\right]^{\frac{1}{\chi}}\cdot$$
$$\left[\int |x|^{\frac{\theta\chi'}{\chi}}\frac{|\nabla\varphi|^{4\chi'}}{\varphi^{3\chi'+\frac{\chi'}{\chi}}}\right]^{\frac{1}{\chi'}} \leqslant \frac{\delta_1^{\chi}}{C_4\chi}\int v^{q+\alpha}b(x)\varphi+$$
$$\frac{[k_1(\alpha)]^{\chi'}}{\chi'\delta_1^{\chi'}}\left[\int |x|^{\frac{\theta\chi'}{\chi}}\frac{|\nabla\varphi|^{4\chi'}}{\varphi^{3\chi'+\frac{\chi'}{\chi}}}\right]$$

b.

$$\left|\frac{1}{2\varepsilon_3^2\alpha}\int u^{\alpha+1}\frac{(\Delta\varphi)^2}{\varphi}\right| \leqslant \frac{1}{2\varepsilon_3^2\alpha}\left[\int \frac{u^{q+\alpha}}{|x|^{\theta}}\varphi\right]^{\frac{1}{\chi}}\cdot$$

$$\left[\int |x|^{\frac{\theta\chi'}{\chi}} \frac{|\Delta\varphi|^{2\chi'}}{\varphi^{\chi'+\frac{\chi'}{\chi}}}\right]^{\frac{1}{\chi'}} \leqslant \frac{\delta_2^{\chi}}{C_3\chi}\int u^{q+\alpha}a(x)\varphi +$$

$$\left[\frac{1}{2\varepsilon_3^2\alpha}\right]^{\chi'} \frac{1}{\chi'\delta_2^{\chi'}}\left[\int |x|^{\frac{\theta\chi'}{\chi}} \frac{|\Delta\varphi|^{2\chi'}}{\varphi^{\chi'+\frac{\chi'}{\chi}}}\right]$$

$$\left|\frac{1}{2\varepsilon_3^2\alpha}\int v^{\alpha+1}\frac{(\Delta\varphi)^2}{\varphi}\right| \leqslant \frac{1}{2\varepsilon_3^2\alpha}\left[\int \frac{v^{q+\alpha}}{|x|^{\theta}}\varphi\right]^{\frac{1}{\chi}} \cdot$$

$$\left[\int |x|^{\frac{\theta\chi'}{\chi}} \frac{|\Delta\varphi|^{2\chi'}}{\varphi^{\chi'+\frac{\chi'}{\chi}}}\right]^{\frac{1}{\chi'}} \leqslant \frac{\delta_2^{\chi}}{C_4\chi}\int v^{q+\alpha}b(x)\varphi +$$

$$\left[\frac{1}{2\varepsilon_3^2\alpha}\right]^{\chi'} \frac{1}{\chi'\delta_2^{\chi'}}\left[\int |x|^{\frac{\theta\chi'}{\chi}} \frac{|\Delta\varphi|^{2\chi'}}{\varphi^{\chi'+\frac{\chi'}{\chi}}}\right]$$

c.

$$\lambda\left|\int u^{\alpha+1}vV_1(x)\varphi\right| \leqslant \lambda\int_{|x|<R_0} u^{\alpha+1}vV_1(x)\varphi +$$

$$C_1\lambda\int_{|x|\geqslant R_0}\frac{u^{\alpha+1}v}{|x|^4}\varphi \leqslant \lambda\int_{|x|<R_0} u^{\alpha+1}vV_1(x)\varphi +$$

$$C_1\lambda\left[\int_{|x|\geqslant R_0}\frac{u^{q+\alpha}}{|x|^{\theta}}\varphi\right]^{\frac{1}{\chi}}\left[\int_{|x|\geqslant R_0} v^{\chi'}|x|^{\frac{\theta\chi'}{\chi}-4\chi'}\varphi\right]^{\frac{1}{\chi'}} \leqslant$$

$$\lambda\int_{|x|<R_0} u^{\alpha+1}vV_1(x)\varphi + \frac{\delta_3^{\chi}}{C_3\chi}\int_{|x|\geqslant R_0} u^{q+\alpha} \cdot$$

$$a(x)\varphi + \frac{\lambda^{\chi'}C_1^{\chi'}}{\chi'\delta_3^{\chi'}}\int_{|x|\geqslant R_0} v^{\chi'}|x|^{\frac{\theta\chi'}{\chi}-4\chi'}\varphi \leqslant$$

$$\lambda\int_{|x|<R_0} u^{\alpha+1}vV_1(x)\varphi + \frac{\delta_3^{\chi}}{C_3\chi}\int_{|x|\geqslant R_0} u^{q+\alpha} \cdot$$

$$a(x)\varphi + \frac{\lambda^{\chi'}C_1^{\chi'}}{\chi'\delta_3^{\chi'}}\left[\int_{|x|\geqslant R_0}\frac{v^{q+\alpha}}{|x|^{\theta}}\varphi\right]^{\frac{1}{y}} \cdot$$

$$\left[\int_{|x|\geqslant R_0}|x|^{\frac{\theta y'}{y}+\frac{\theta\chi'}{\chi}y'-4y'\chi'}\varphi\right]^{\frac{1}{y'}} \leqslant$$

$$\lambda\int_{|x|<R_0} u^{\alpha+1}vV_1(x)\varphi + \frac{\delta_3^{\chi}}{C_3\chi}\int_{|x|\geqslant R_0} u^{q+\alpha} \cdot$$

$$a(x)\varphi + \frac{\lambda^{\chi'}C_1^{\chi'}\delta_4^{y}}{\chi'\delta_3^{\chi'}C_4 y}\int_{|x|\geqslant R_0} v^{q+\alpha}b(x)\varphi +$$

$$\frac{\lambda^{\chi'}C_1^{\chi'}}{\chi'\delta_3^{\chi'}y'\delta^{y'}{}_4}\int_{|x|\geqslant R_0}|x|^{\frac{\theta y'}{y}+\frac{\theta\chi'}{\chi}y'-4y'\chi'}\varphi$$

$$\lambda\left|\int v^{\alpha+1}uV_2(x)\varphi\right| \leqslant \lambda\int_{|x|<R_0} v^{\alpha+1}uV_2(x)\varphi +$$

$$C_2\lambda\int_{|x|\geqslant R_0}\frac{v^{\alpha+1}u}{|x|^4}\varphi \leqslant \lambda\int_{|x|<R_0} v^{\alpha+1}uV_2(x)\varphi +$$

$$C_2\lambda\left[\int_{|x|\geqslant R_0}\frac{v^{q+\alpha}}{|x|^{\theta}}\varphi\right]^{\frac{1}{\chi}}\left[\int_{|x|\geqslant R_0} u^{\chi'}|x|^{\frac{\theta\chi'}{\chi}-4\chi'}\varphi\right]^{\frac{1}{\chi'}} \leqslant$$

$$\lambda\int_{|x|<R_0} v^{\alpha+1}uV_2(x)\varphi + \frac{\delta_3^{\chi}}{C_4\chi}\int_{|x|\geqslant R_0} v^{q+\alpha} \cdot$$

$$b(x)\varphi + \frac{\lambda^{\chi'}C_2^{\chi'}}{\chi'\delta_3^{\chi'}}\int_{|x|\geqslant R_0} u^{\chi'}|x|^{\frac{\theta\chi'}{\chi}-4\chi'}\varphi \leqslant$$

$$\lambda\int_{|x|<R_0} v^{\alpha+1}uV_2(x)\varphi + \frac{\delta_3^{\chi}}{C_4\chi}\int_{|x|\geqslant R_0} v^{q+\alpha} \cdot$$

$$b(x)\varphi + \frac{\lambda^{\chi'}C_2^{\chi'}}{\chi'\delta_3^{\chi'}}\left[\int_{|x|\geqslant R_0}\frac{u^{q+\alpha}}{|x|^{\theta}}\varphi\right]^{\frac{1}{y}} \cdot$$

$$\left[\int_{|x|\geqslant R_0}|x|^{\frac{\theta y'}{y}+\frac{\theta\chi'}{\chi}y'-4y'\chi'}\varphi\right]^{\frac{1}{y'}} \leqslant$$

$$\lambda\int_{|x|<R_0} v^{\alpha+1}uV_2(x)\varphi + \frac{\delta_3^{\chi}}{C_4\chi}\int_{|x|\geqslant R_0} v^{q+\alpha} \cdot$$

$$b(x)\varphi + \frac{\lambda^{\chi'}C_2^{\chi'}\delta_4^{y}}{\chi'\delta_3^{\chi'}C_3 y}\int_{|x|\geqslant R_0} u^{q+\alpha}a(x)\varphi +$$

$$\frac{\lambda^{\chi'}C_2^{\chi'}}{\chi'\delta_3^{\chi'}y'\delta_4^{y'}}\int_{|x|\geqslant R_0}|x|^{\frac{\theta y'}{y}+\frac{\theta\chi'}{\chi}y'-4y'\chi'}\varphi$$

将 a,b,c 代入式(20)，有

$$C_4(\alpha)\int u^{q+\alpha}a(x)\varphi + k_3(\alpha)\int\alpha(\alpha-1)u^{\alpha-2} \cdot$$

$$\Delta u|\nabla u|^2\varphi + k_2(\alpha)\int\alpha u^{\alpha-1}(\Delta u)^2\varphi \leqslant$$

$$\lambda\int_{|x|<R_0} u^{\alpha+1}vV_1(x)\varphi + C_1(\alpha)\int_{A_R}|x|^{\frac{\theta\chi'}{\chi}} \cdot$$

$$\frac{|\nabla\varphi|^{4\chi'}}{\varphi^{3\chi'+\frac{\chi'}{\chi}}} + C_2(\alpha)\int_{A_R}|x|^{\frac{\theta\chi'}{\chi}}\frac{|\Delta\varphi|^{2\chi'}}{\varphi^{\chi'+\frac{\chi'}{\chi}}} +$$

$$C_3(\alpha)\int_{|x|\geqslant R_0} v^{q+\alpha}b(x)\varphi +$$

$$C_5(\alpha)\int_{|x|\geqslant R_0}|x|^{\frac{\theta y'}{y}+\frac{\theta\chi'}{\chi}y'-4y'\chi'}\varphi \tag{21}$$

$$C_6(\alpha)\int v^{q+\alpha}b(x)\varphi + k_3(\alpha)\int\alpha(\alpha-1)v^{\alpha-2} \cdot$$

$$\Delta v\,|\nabla v|^2\varphi + k_2(\alpha)\int\alpha v^{\alpha-1}(\Delta v)^2\varphi \leqslant$$

$$\lambda\int_{|x|<R_0} v^{\alpha+1}uV_2(x)\varphi + C_1(\alpha)\int_{A_R}|x|^{\frac{\theta\chi'}{\chi}} \cdot$$

$$\frac{|\nabla\varphi|^{4\chi'}}{\varphi^{3\chi'+\frac{\chi'}{\chi}}} + C_2(\alpha)\int_{A_R}|x|^{\frac{\theta\chi'}{\chi}}\frac{|\Delta\varphi|^{2\chi'}}{\varphi^{\chi'+\frac{\chi'}{\chi}}} +$$

$$C_7(\alpha)\int_{|x|\geqslant R_0} u^{q+\alpha}a(x)\varphi +$$

$$C_8(\alpha)\int_{|x|\geqslant R_0}|x|^{\frac{\theta y'}{y}+\frac{\theta\chi'}{\chi}y'-4y'\chi'}\varphi \tag{22}$$

其中

$$C_3(\alpha) = \frac{\lambda^{\chi'}C_1^{\chi'}\delta_4^{y}}{\chi'\delta_3^{\chi'}yC_4}$$

$$C_4(\alpha) = 1 - \frac{\delta_1^{\chi}}{C_3\chi} - \frac{\delta_2^{\chi}}{C_3\chi} - \frac{\delta_3^{\chi}}{C_3\chi}$$

$$C_5(\alpha) = \frac{\lambda^{\chi'}C_1^{\chi'}}{\chi'\delta_3^{\chi'}y'\delta_4^{y'}}$$

$$C_6(\alpha) = 1 - \frac{\delta_1^{\chi}}{C_4\chi} - \frac{\delta_2^{\chi}}{C_4\chi} - \frac{\delta_3^{\chi}}{C_4\chi}$$

$$C_7(\alpha) = \frac{\lambda^{\chi'}C_2^{\chi'}\delta_4^{y}}{\chi'\delta_3^{\chi'}C_3 y},\quad C_8(\alpha) = \frac{\lambda^{\chi'}C_2^{\chi'}}{\chi'\delta_3^{\chi'}y'\delta_4^{y'}}$$

将式(21)与式(22)相加，有

$$[C_4(\alpha) - C_7(\alpha)]\int u^{q+\alpha}a(x)\varphi + [C_6(\alpha) -$$

$$C_3(\alpha)]\int v^{q+\alpha}b(x)\varphi + k_3(\alpha)\left[\int\alpha(\alpha-1)u^{\alpha-2} \cdot\right.$$

$$\left.\Delta u\,|\nabla u|^2\varphi + \int\alpha(\alpha-1)v^{\alpha-2}\Delta v\,|\nabla v|^2\varphi\right] +$$

$$k_2(\alpha)\left[\int \alpha u^{\alpha-1}(\Delta u)^2\varphi+\int \alpha v^{\alpha-1}(\Delta v)^2\varphi\right]\leqslant$$

$$\lambda\left[\int_{|x|<R_0} u^{\alpha+1}vV_1(x)\varphi+\int_{|x|<R_0} v^{\alpha+1}u\cdot\right.$$

$$\left.V_2(x)\varphi\right]+2C_1(\alpha)\int_{A_R}|x|^{\frac{\theta\chi'}{\chi}}\frac{|\nabla\varphi|^{4\chi'}}{\varphi^{3\chi'+\frac{\chi'}{\chi}}}+$$

$$2C_2(\alpha)\int_{A_R}|x|^{\frac{\theta\chi'}{\chi}}\frac{|\Delta\varphi|^{2\chi'}}{\varphi^{\chi'+\frac{\chi'}{\chi}}}+[C_5(\alpha)+$$

$$C_8(\alpha)]\int_{|x|\geqslant R_0}|x|^{\frac{\theta y'}{y}+\frac{\theta\chi'}{\chi}y'-4y'\chi'}\varphi$$

取足够小的 $\varepsilon_1,\varepsilon_2,\varepsilon_3,\delta_1,\delta_2,\delta_4$，固定 δ_3，使得 $C_4(\alpha)-C_7(\alpha)$，$C_6(\alpha)-C_3(\alpha)$，$k_2(\alpha)$，$k_3(\alpha)$为正，令 $R\to\infty$，由推论 1 及单调收敛定理得

$$\int u^{q+\alpha}a(x)\varphi+\int v^{q+\alpha}b(x)\varphi<C$$

因为，C 与 φ 无关，所以，有

$$u^{q+\alpha}a(x)+v^{q+\alpha}b(x)\in L^1(R^N)$$

即得 $u^{q+\alpha}a(x)\in L^1(R^N)$，$v^{q+\alpha}b(x)\in L^1(R^N)$.

引理 4[10] 令 $r>0$，$u,v\in W^{2,2}(B_r)$，使得在 B_{2r}上，几乎处处有：$u\geqslant 0$，$v\geqslant 0$，$\Delta u\geqslant 0$，$\Delta v\geqslant 0$，则有

$$\|u\|_{L^\infty(B_r)}+\|v\|_{L^\infty(B_r)}\leqslant$$

$$C\left[\left(\fint_{B_{2r}}u^2\mathrm{d}x\right)^{\frac{1}{2}}+\left(\fint_{B_{2r}}v^2\mathrm{d}x\right)^{\frac{1}{2}}\right]$$

其中，常数 C 与 r 无关.

3 主要结论

现证明定理 1.

定理 1 令 u,v 是不等式组(1)的弱解，且满足定义 1 及在 R^N 上，几乎处处有：$u\geqslant 0$，$v\geqslant 0$，$\Delta u\geqslant 0$，$\Delta v\geqslant 0$，则有，$u=v\equiv 0$，即不等式组(1)不存在整体解.

证明 取 $\alpha>2$，使得

$$2<q<\frac{N-\theta+\alpha(4-\theta)}{N-4},\sigma=\frac{q+\alpha}{2}$$

易证 u^σ，v^σ 满足引理 4 的假设，$\forall\, r>0$，有

$$\|u^\sigma\|_{L^\infty(B_r)}+\|v^\sigma\|_{L^\infty(B_r)}\leqslant$$

$$C\left[\left(\fint B_{2r}u^{2\sigma}\mathrm{d}x\right)^{\frac{1}{2}}+\left(\fint_{B_{2r}}v^{2\sigma}\mathrm{d}x\right)^{\frac{1}{2}}\right]\leqslant$$

$$C\left[\left(\frac{1}{2^Nr^N}\int_{B_{2r}}u^{2\sigma}\mathrm{d}x\right)^{\frac{1}{2}}+\left(\frac{1}{2^Nr^N}\int_{B_{2r}}v^{2\sigma}\mathrm{d}x\right)^{\frac{1}{2}}\right]\leqslant$$

$$C\left[\left(\frac{1}{r^{N-\theta}}\int_{B_{2r}}\frac{u^{2\sigma}}{r^\theta}\mathrm{d}x\right)^{\frac{1}{2}}+\left(\frac{1}{r^{N-\theta}}\int_{B_{2r}}\frac{v^{2\sigma}}{r^\theta}\mathrm{d}x\right)^{\frac{1}{2}}\right]\leqslant$$

$$C\left[\left(\frac{1}{r^{N-\theta}}\int_{B_{2r}}\frac{u^{2\sigma}}{|x|^\theta}\mathrm{dx}\right)^{\frac{1}{2}}+\left(\frac{1}{r^{N-\theta}}\int_{B_{2r}}\frac{v^{2\sigma}}{|x|^\theta}\mathrm{d}x\right)^{\frac{1}{2}}\right]\leqslant$$

$$C\left[\left(\frac{1}{C_3r^{N-\theta}}\int_{B_{2r}}a(x)u^{2\sigma}\right)^{\frac{1}{2}}+\left(\frac{1}{C_4r^{N-\theta}}\int_{B_{2r}}b(x)v^{2\sigma}\right)^{\frac{1}{2}}\right]$$

因为，$u^{q+\alpha}a(x)\in L^1(R^N)$，$v^{q+\alpha}b(x)\in L^1(R^N)$，所以

$$a(x)u^{2\sigma}\in L^1(R^N),\ b(x)v^{2\sigma}\in L^1(R^N)$$

因为，$N>4>\theta$，所以，当 $r\to\infty$时，有 $\|u^\sigma\|_{L^\infty(B_r)}+\|v^\sigma\|_{L^\infty(B_r)}\leqslant 0$，则 $u=v\equiv 0$.

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（编辑：石　瑛）