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一类具有非瞬时脉冲的分数阶微分方程积分边值问题解的存在性

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摘要： 研究了一类具有积分边界条件的非瞬时脉冲分数阶微分方程边值问题.根据非瞬时脉冲条件和边界条件的特点，针对非线性项不同的控制条件，建立了边值问题解的存在性和唯一性的多个定理，并运用不动点定理证明了所得结论的正确性.

关键词： 非瞬时脉冲；积分边值问题；Caputo 分数阶导数；不动点定理
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Existence of Solutions for Integral Boundary Value Problems of Fractional Differential Equations with Non-instantaneous Impulses

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Abstract：A class of boundary value problems of non-instantaneous impulses fractional differential equations with integral boundary conditions was studied. According to the conditions of non-instantaneous impulses and the properties of boundary conditions, multiple theorems for the existence and uniqueness of boundary value problem solutions were established under different control conditions of nonlinear terms, and also the correctness of the obtained solutions was proved by using the fixed point theorem.

Keywords：*non-instantaneous impulse*；*integral boundary value problem*；*Caputo fractional derivative*；*fixed point theorem*

1　问题的提出

近年来，分数阶微分方程在现代科学技术领域的应用日益广泛，其理论研究也取得了巨大的进展[1-8].作为刻画突变现象的脉冲微分方程在电子技术及通讯工程等方面发挥了巨大的作用，瞬时脉冲理论也受到人们的关注[9-13].在生物技术及医药工程领域存在着大量的非瞬时脉冲现象[14-20]，对这

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类现象进行研究具有重要意义.本文研究一类具有非瞬时脉冲的分数阶微分方程积分边值问题

$$\begin{cases} {}^{c}\mathrm{D}_{0^+}^{\alpha} u(t) = f(t,u(t)),\ t\in(s_k,t_{k+1}], \\ \quad k=0,1,\cdots,m \\ u(t)=h_k(t),\quad t\in(t_k,s_k] \\ u'(s_k)=h_k'(s_k),\ k=1,2,\cdots,m \\ u(0)=0,\ u(1)=\int_0^1 g(s)u(s)\mathrm{d}s \end{cases} \tag{1}$$

式中:${}^{c}\mathrm{D}_{0^+}^{\alpha}$ 为标准的 Caputo 分数阶导数,$\alpha\in(1,2)$;$J=[0,1]$,$0=s_0<t_1<s_1<t_2<\cdots<s_m<t_{m+1}=1$.令 $J_0=J\backslash\{t_1,t_2,\cdots,t_m\}$,$f:J\times\mathbb{R}\to\mathbb{R}$ 为连续函数,且 h_k 在 J 上具有一阶连续导数,其中,$k=1,2,\cdots,m$.

定义空间:$PC(J,\mathbb{R}):=\{u:J\to\mathbb{R}\mid u\in C(J_0,\mathbb{R}),u(t_k^-),u(t_k^+)$ 存在,且 $u(t_k^-)=u(t_k),k=1,2,\cdots,m\}$,取范数 $\|u\|_{PC}=\sup_{t\in[0,1]}|u(t)|$,则 $PC(J,\mathbb{R})$ 为 Banach 空间.

令 $\rho=\int_0^{t_1} sg(s)\mathrm{d}s$,并假设 H0 总是成立

H0 $g\in L^1(J)$,且 $\rho\neq 0$.

2 预备知识与引理

有关分数阶微积分的定义可参见文献[1-4].

定义 1 设 $u\in PC(J,\mathbb{R})$,若 u 满足边值问题(1)中的各等式,则称 u 是边值问题(1)的一个解.

引理 1[4] 假设 $y\in L[0,1]$,且 $1<\alpha<2$,那么,Cauchy 问题

$$\begin{cases} {}^{c}\mathrm{D}_{0^+}^{\alpha} u(t)=y(t),\quad t\in[0,1] \\ u(\xi)=u_0,\ 0<\xi<1 \end{cases} \tag{2}$$

有解,且以下列形式给出:

$$u(t)=u_0-I_{0^+}^{\alpha}y(\xi)+I_{0^+}^{\alpha}y(t)+d(t-\xi),d\in\mathbb{R}$$

推论 1 假设 $y\in L[0,1]$,且 $1<\alpha<2$,那么,Cauchy 问题

$$\begin{cases} {}^{c}\mathrm{D}_{0^+}^{\alpha} u(t)=y(t),t\in[0,1] \\ u(\xi)=u_0,\ u'(\xi)=u_1,\ 0<\xi<1 \end{cases} \tag{3}$$

存在唯一解

$$u(t)=u_0-I_{0^+}^{\alpha}y(\xi)+I_{0^+}^{\alpha}y(t)+ (u_1-I_{0^+}^{\alpha-1}y(\xi))(t-\xi)$$

引理 2 对任意的 $y\in L^1[0,1]$,且 H0 成立,则边值问题

$$\begin{cases} {}^{c}\mathrm{D}_{0^+}^{\alpha} u(t)=y(t),\ t\in(s_k,t_{k+1}], \\ \quad k=0,1,\cdots,m \\ u(t)=h_k(t),\ t\in(t_k,s_k] \\ u'(s_k)=h_k'(s_k),\ k=1,2,\cdots,m \\ u(0)=0,\ u(1)=\int_0^1 g(s)u(s)\mathrm{d}s \end{cases} \tag{4}$$

存在唯一解

$$u(t)=v(t)+G(t) \tag{5}$$

其中

$$v(t)=\begin{cases} \int_0^1\Bigg[\frac{\chi(t,s)(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t}{\rho}\Big(\omega(s)-\chi(s_m,s)\frac{(1-s_m)(s_m-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big)+ \\ \quad \frac{t}{\rho}\sum_{k=1}^{m}\frac{\chi(s_k,s)\int_{s_k}^{t_{k+1}}(\tau-s_k)g(\tau)\mathrm{d}\tau}{\Gamma(\alpha-1)}(s_k-s)^{\alpha-2}\Bigg]y(s)\mathrm{d}s,\quad t\in[0,t_1] \\ 0,\quad t\in(t_k,s_k],k=1,2,\cdots,m \\ \int_0^1\Bigg[\frac{\chi(t,s)(t-s)^{\alpha-1}}{\Gamma(\alpha)}-\chi(s_k,s)\Big(\frac{(s_k-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(t-s_k)(s_k-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big)\Bigg]y(s)\mathrm{d}s, \\ \quad t\in(s_k,t_{k+1}],k=1,2,\cdots,m \end{cases} \tag{6}$$

$$G(t)=\begin{cases} \frac{t}{\rho}\Big(-\sum_{k=1}^{m}\int_{t_k}^{s_k}g(s)h_k(s)\mathrm{d}s-\sum_{k=1}^{m}\int_{s_k}^{t_{k+1}}g(s)(h_k(s_k)+h_k'(s_k)(s-s_k))\mathrm{d}s+ \\ \quad h_m(s_m)+h_m'(s_m)(1-s_m)\Big),\quad t\in[0,t_1] \\ h_k(t),\quad t\in(t_k,s_k],\ k=1,2,\cdots,m \\ h_k(s_k)+h_k'(s_k)(t-s_k),\ t\in(s_k,t_{k+1}],\ k=1,2,\cdots,m \end{cases} \tag{7}$$

$$\omega(s)=\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}-\frac{\chi(s_m,s)(s_m-s)^{\alpha-1}}{\Gamma(\alpha)}-\sum_{k=0}^{m}\chi(t_{k+1},s)I_{t_{k+1}^-}^{\alpha}g(s)+\sum_{k=1}^{m}\chi(s_k,s)\cdot \left(I_{s_k^-}^{\alpha}g(s)+\frac{\int_{s_k}^{t_{k+1}}g(\tau)\mathrm{d}\tau}{\Gamma(\alpha)}(s_k-s)^{\alpha-1}\right),\qquad \chi(t,s)=\begin{cases}1,& s\leqslant t\\ 0,& t<s\end{cases} \tag{8}$$

证明　当 $t\in[0,t_1]$时，由 ${}^{c}\mathrm{D}_{0^+}^{\alpha}u(t)=y(t)$，可得 $u(t)=I_{0^+}^{\alpha}y(t)+c_0+c_1t$，其中，$c_i\in\mathbb{R}$，$i=0,1$.

根据边界条件 $u(0)=0$，可得 $c_0=0$，故

$$u(t)=I_{0^+}^{\alpha}y(t)+c_1t \tag{9}$$

当 $t\in(t_1,s_1]$时，有 $u(t)=h_1(t)$，$u'(t)=h_1'(t)$.

考虑 Cauchy 问题

$$\begin{cases}{}^{c}\mathrm{D}_{0^+}^{\alpha}u(t)=y(t), t\in[0,1]\\ u(s_1)=h_1(s_1), u'(s_1)=h_1'(s_1)\end{cases} \tag{10}$$

当 $t\in(s_1,t_2]$时，有 $u(t)=I_{0^+}^{\alpha}y(t)-I_{0^+}^{\alpha}y(s_1)-(t-s_1)I_{0^+}^{\alpha-1}y(s_1)+h_1(s_1)+h_1'(s_1)\cdot(t-s_1)$；当 $t\in(t_k,s_k]$，$k=1,2,\cdots,m$ 时，有 $u(t)=h_k(t)$.

类似地，当 $t\in(s_k,t_{k+1}]$，$k=1,2,\cdots,m$ 时，

$$u(t)=I_{0^+}^{\alpha}y(t)-I_{0^+}^{\alpha}y(s_k)-(t-s_k)I_{0^+}^{\alpha-1}y(s_k)+h_k(s_k)+h_k'(s_k)(t-s_k)$$

因此，

$$\begin{aligned}u(1)={}&I_{0^+}^{\alpha}y(1)-I_{0^+}^{\alpha}y(s_m)-(1-s_m)I_{0^+}^{\alpha-1}y(s_m)+h_m(s_m)+h_m'(s_m)(1-s_m)=\\&\int_0^1\left[\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}-\chi(s_m,s)\left(\frac{(s_m-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s_m)(s_m-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)\right]y(s)\mathrm{d}s+\\&h_m(s_m)+h_m'(s_m)(1-s_m)\end{aligned}$$

另一方面，

$$\begin{aligned}\int_0^1 g(s)u(s)\mathrm{d}s={}&\sum_{k=0}^{m}\int_{s_k}^{t_{k+1}}g(s)I_{0^+}^{\alpha}y(s)\mathrm{d}s+\sum_{k=1}^{m}\int_{t_k}^{s_k}g(s)h_k(s)\mathrm{d}s+c_1\int_0^{t_1}sg(s)\mathrm{d}s-\\&\sum_{k=1}^{m}\left(\int_{s_k}^{t_{k+1}}g(s)I_{0^+}^{\alpha}y(s_k)\mathrm{d}s+\int_{s_k}^{t_{k+1}}(s-s_k)g(s)I_{0^+}^{\alpha-1}\cdot y(s_k)\mathrm{d}s-\int_{s_k}^{t_{k+1}}g(s)(h_k(s_k)+h_k'(s_k)(s-s_k))\mathrm{d}s\right)=\\&\sum_{k=0}^{m}\left(\int_0^{t_{k+1}}y(s)I_{t_{k+1}^-}^{\alpha}g(s)\mathrm{d}s-\int_0^{s_k}y(s)I_{s_k^-}^{\alpha}g(s)\mathrm{d}s\right)+\sum_{k=1}^{m}\int_{t_k}^{s_k}g(s)h_k(s)\mathrm{d}s+c_1\int_0^{t_1}sg(s)\mathrm{d}s-\\&\sum_{k=1}^{m}\left|\frac{\int_{s_k}^{t_{k+1}}g(\tau)\mathrm{d}\tau}{\Gamma(\alpha)}\int_0^{s_k}(s_k-s)^{\alpha-1}y(s)\mathrm{d}s+\frac{\int_{s_k}^{t_{k+1}}(\tau-s_k)g(\tau)\mathrm{d}\tau}{\Gamma(\alpha-1)}\int_0^{s_k}(s_k-s)^{\alpha-2}y(s)\mathrm{d}s-\right.\\&\left.\int_{s_k}^{t_{k+1}}g(s)(h_k(s_k)+h_k'(s_k)(s-s_k))\mathrm{d}s\right|=\\&\int_0^1\left(\sum_{k=0}^{m}\chi(t_{k+1},s)I_{t_{k+1}^-}^{\alpha}g(s)-\sum_{k=1}^{m}\chi(s_k,s)\left(I_{s_k^-}^{\alpha}g(s)+\frac{\int_{s_k}^{t_{k+1}}(\tau-s_k)g(\tau)\mathrm{d}\tau}{\Gamma(\alpha-1)}(s_k-s)^{\alpha-2}+\frac{\int_{s_k}^{t_{k+1}}g(\tau)\mathrm{d}\tau}{\Gamma(\alpha)}(s_k-s)^{\alpha-1}\right)\right)y(s)\mathrm{d}s+\\&\sum_{k=1}^{m}\int_{t_k}^{s_k}g(s)h_k(s)\mathrm{d}s+c_1\int_0^{t_1}sg(s)\mathrm{d}s+\\&\sum_{k=1}^{m}\int_{s_k}^{t_{k+1}}g(s)(h_k(s_k)+h_k'(s_k)(s-s_k))\mathrm{d}s\end{aligned}$$

由边界条件 $u(1)=\int_0^1 g(s)u(s)\mathrm{d}s$，可得

$$\begin{aligned}c_1=\frac{1}{\rho}\Bigg[&\int_0^1\left(\omega(s)-\chi(s_m,s)\frac{(1-s_m)(s_m-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\sum_{k=1}^{m}\chi(s_k,s)\frac{\int_{s_k}^{t_{k+1}}(\tau-s_k)g(\tau)\mathrm{d}\tau}{\Gamma(\alpha-1)}(s_k-s)^{\alpha-2}\right)y(s)\mathrm{d}s-\\&\sum_{k=1}^{m}\int_{t_k}^{s_k}g(s)h_k(s)\mathrm{d}s-\sum_{k=1}^{m}\int_{s_k}^{t_{k+1}}g(s)(h_k(s_k)+\\&h_k'(s_k)(s-s_k))\mathrm{d}s+h_m(s_m)+h_m'(s_m)(1-s_m)\Bigg]\end{aligned}$$

将上式代入式(9)，可得式(5)成立，即引理 2 得证.

经过化简与计算，易得下面的引理 3 和引理 4.

引理 3　对任意的 $s\in J$，有以下不等式成立：

a. $|\omega(s)|\leqslant\dfrac{2(m+1)\|g\|_{L^1}+2}{\Gamma(\alpha)}$；

b. $\displaystyle\int_0^1|\omega(s)|\mathrm{d}s\leqslant\frac{1+s_m^{\alpha}+2\sum_{k=0}^{m}t_{k+1}^{\alpha}\|g\|_{L^1}}{\Gamma(\alpha+1)}$.

为了方便起见，记

$$N_0=\frac{|\rho|+1+s_m^{\alpha}+2\sum_{k=0}^{m}t_{k+1}^{\alpha}\|g\|_{L^1}}{|\rho|\Gamma(\alpha+1)}+\frac{(1-s_m)s_m^{\alpha-1}+s_m^{\alpha-1}\|g\|_{L^1}}{|\rho|\Gamma(\alpha)}$$

$$N_1=\frac{1+2s_m^{\alpha-1}}{\Gamma(\alpha)}$$

$$N_2=\frac{(2M_h+p)\|g\|_{L^1}+M_h+p}{|\rho|}$$

$$N_3=\frac{(2m+2+s_m^{\alpha-1})\|g\|_{L^1}+2+|\rho|+s_m^{\alpha-1}}{|\rho|\Gamma(\alpha)}$$

$$M_h = \max_{1\leqslant k\leqslant m}\{\sup_{t\in[t_k,s_k]}|h_k(t)|\}$$

$$p = \max_{1\leqslant k\leqslant m}\{|h'_k(s_k)|\}$$

根据式(7),容易证明引理4成立.

引理4 对任意的 $t\in J$,不等式 $|G(t)|\leqslant N_2$ 成立.

定义算子 $A:PC(J,\mathbb{R})\to PC(J,\mathbb{R})$.

$$Au(t)=\begin{cases}\int_0^1\Big[\dfrac{\chi(t,s)(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{t}{\rho}\Big(\omega(s)-\chi(s_m,s)\dfrac{(1-s_m)(s_m-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\\ \quad\displaystyle\sum_{k=1}^{m}\dfrac{\chi(s_k,s)\int_{s_k}^{t_{k+1}}(\tau-s_k)g(\tau)\mathrm{d}\tau}{\Gamma(\alpha-1)}(s_k-s)^{\alpha-2}\Big)\Big]f(s,u(s))\mathrm{d}s,\ t\in[0,t_1]\\ 0,\ t\in(t_k,s_k],\ k=1,2,\cdots,m\\ \int_0^1\Big[\dfrac{\chi(t,s)(t-s)^{\alpha-1}}{\Gamma(\alpha)}-\chi(s_k,s)\Big(\dfrac{(s_k-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(t-s_k)(s_k-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big)\Big]\cdot\\ \quad f(s,u(s))\mathrm{d}s,\ t\in(s_k,t_{k+1}],\ k=1,2,\cdots,m\end{cases}\tag{11}$$

记

$$Tu(t)=Au(t)+G(t)\tag{12}$$

引理5 若假设H0成立,则算子 $T:PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ 全连续.

证明 首先,证明 T 为连续算子.

设 $u_n,u\in PC(J,\mathbb{R}),n=1,2,\cdots$, 且 $\|u_n-u\|_{PC}\to 0(n\to\infty)$, 即对任意的 $t\in J$,有 $u_n(t)\to u(t)(n\to\infty)$,由于 f 是连续函数,则 $|f(s,u_n(s))-f(s,u(s))|\to 0,n\to\infty$.

故根据Lebesgue控制收敛定理,可得

$$\int_0^1|f(s,u_n(s))-f(s,u(s))|\mathrm{d}s\to 0,n\to\infty$$

因此,当 $t\in[0,t_1]$时,结合引理3的a,可得

$$|Tu_n(t)-Tu(t)|\leqslant\Big(\frac{|\rho|+2(m+1)\|g\|_{L^1}+2}{|\rho|\Gamma(\alpha)}+\frac{s_m^{\alpha-1}+s_m^{\alpha-1}\|g\|_{L^1}}{|\rho|\Gamma(\alpha)}\Big)\int_0^1\Big|f(s,u_n(s))-f(s,u(s))\Big|\mathrm{d}s\leqslant N_3\int_0^1\Big|f(s,u_n(s))-f(s,u(s))\Big|\mathrm{d}s\to 0,n\to\infty$$

当 $t\in(t_k,s_k],k=1,2,\cdots,m$ 时,$|Tu_n(t)-Tu(t)|=0$.

类似可得,当 $t\in(s_k,t_{k+1}],k=1,2,\cdots,m$ 时,

$$|Tu_n(t)-Tu(t)|\leqslant\frac{1+2s_m^{\alpha-1}}{\Gamma(\alpha)}\cdot\int_0^1|f(s,u_n(s))-f(s,u(s))|\mathrm{d}s\to 0,n\to\infty$$

即有 $\|Tu_n-Tu\|_{PC}\to 0(n\to\infty)$, 因此,$T$ 连续.

其次,证明 T 是紧的.

设 $B\subset PC(J,\mathbb{R})$ 为有界集,则存在 $r>0$,使得对任意的 $u\in B$,有 $\|u\|_{PC}\leqslant r$,记 $M_f=\max_{(t,u)\in[0,1]\times[-r,r]}|f(t,u)|$,则当 $t\in[0,t_1]$时,结合引理3的a与引理4,可得

$$|Tu(t)|\leqslant|Au(t)|+|G(t)|\leqslant\int_0^1\Big[\frac{1}{\Gamma(\alpha)}+\frac{|\omega(s)|}{|\rho|}+\Big(\frac{\chi(s_m,s)(s_m-s)^{\alpha-2}}{|\rho|\Gamma(\alpha-1)}+\sum_{k=1}^{m}\frac{\chi(s_k,s)\int_{s_k}^{t_{k+1}}|g(\tau)|\mathrm{d}\tau}{|\rho|\Gamma(\alpha-1)}(s_k-s)^{\alpha-2}\Big)\Big]\cdot|f(s,u(s))|\mathrm{d}s+N_2\leqslant$$

$$M_f\int_0^1\Big[\frac{|\rho|+2(m+1)\|g\|_{L^1}+2}{|\rho|\Gamma(\alpha)}+\Big(\chi(s_m,s)\frac{(s_m-s)^{\alpha-2}}{|\rho|\Gamma(\alpha-1)}+\frac{\chi(s_k,s)\|g\|_{L^1}}{|\rho|\Gamma(\alpha-1)}\cdot(s_k-s)^{\alpha-2}\Big)\Big]\mathrm{d}s+N_2\leqslant M_fN_3+N_2$$

当 $t\in(t_k,s_k],k=1,2,\cdots,m$ 时,$|Tu(t)|=|h_k(t)|\leqslant M_h$.

当 $t\in(s_k,t_{k+1}],k=1,2,\cdots,m$ 时,

$$|Tu(t)|\leqslant|Au(t)|+|G(t)|\leqslant\int_0^1\Big(\frac{1+s_m{}^{\alpha-1}}{\Gamma(\alpha)}+\frac{\chi(s_k,s)(s_k-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big)\cdot|f(s,u(s))|\mathrm{d}s+N_2\leqslant M_fN_1+N_2$$

因此,$T(B)$一致有界.

对任意的 $\varepsilon>0$,记 $\delta_1=\dfrac{\varepsilon}{2(M_fN_3+N_2)}$,$\delta_2=\dfrac{\Gamma(\alpha)\varepsilon}{2(2M_f+p\Gamma(\alpha))}$,对 $k=1,2,\cdots,m$,由于 $h_k(t)$ 在 J 上连续,所以,$h_k(t)$在 J 上一致连续,故存在常数 $\delta_3>0$, 使得当 $t'_1,t'_2\in J$,且 $t'_1<t'_2$,$|t'_1-t'_2|<\delta_3$ 时,有 $|h_k(t'_1)-h_k(t'_2)|<\varepsilon$. 又因为,$(t-s)^{\alpha-1}$在$(t,s)\in[0,1]\times[0,1]$连续,所以,存在常数 $0<\delta\leqslant\min\{\delta_1,\delta_2,\delta_3\}$,对任意 $t'_1,t'_2,s'_1,s'_2\in J$, $|t'_1-t'_2|<\delta$, 有 $|(t'_1-s)^{\alpha-1}-(t'_2-s)^{\alpha-1}|<\dfrac{\Gamma(\alpha)\varepsilon}{2M_f}$.因此,当 $t'_1<t'_2\in[0,t_1]$,且

$|t_1'-t_2'|<\delta$ 时,结合引理 3 的 a 与引理 4,可得

$$|Tu(t_1')-Tu(t_2')|\leqslant$$

$$M_f\Big(\int_0^{t_1'}\frac{|(t_1'-s)^{\alpha-1}-(t_2'-s)^{\alpha-1}|}{\Gamma(\alpha)}ds+\int_{t_1'}^{t_2'}\frac{(t_2'-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\frac{\delta_1(|\rho|+(2m+3)\|g\|_{L^1}+3)}{|\rho|\Gamma(\alpha)}\Big)+$$

$$\frac{\delta_1((2M_h+p)\|g\|_{L^1}+M_h+p)}{|\rho|}\leqslant$$

$$\frac{M_f}{\Gamma(\alpha)}\frac{\Gamma(\alpha)\varepsilon}{2M_f}+\delta_1(M_fN_3+N_1)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}<\varepsilon$$

当 $t_1'<t_2'\in(t_k,s_k]$, $k=1,2,\cdots,m$,且 $|t_1'-t_2'|<\delta$ 时,

$$|Tu(t'_1)-Tu(t'_2)|=|h_k(t'_1)-h_k(t'_2)|<\varepsilon$$

类似可得,当 $t_1'<t_2'\in(s_k,t_{k+1}]$, $k=1,2,\cdots,m$,且 $|t_1'-t_2'|<\delta$ 时,

$$|Tu(t_1')-Tu(t_2')|\leqslant$$

$$M_f\Big(\int_0^{t_1'}\frac{|(t_1'-s)^{\alpha-1}-(t_2'-s)^{\alpha-1}|}{\Gamma(\alpha)}ds+\int_{t_1'}^{t'_2}\frac{(t'_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\delta_2\int_0^{s_k}\frac{(s_k-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\Big)+\delta_2 p\leqslant$$

$$\frac{M_f}{\Gamma(\alpha)}\frac{\Gamma(\alpha)\varepsilon}{2M_f}+\delta_2\Big(\frac{2M_f+p\Gamma(\alpha)}{\Gamma(\alpha)}\Big)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}<\varepsilon$$

即 $T(B)$ 等度连续. 由 Arzela-Ascoli 定理可知 T 是紧的.

综合以上讨论,T 是全连续算子.

3　边值问题解的存在性与唯一性

假设:

H1　存在非负函数 $a_0,a_1\in C[0,1]$,常数 $\sigma>0$,使得对任意的 $t\in J$ 及任意的 $x\in\mathbb{R}$,有

$$|f(t,x)|\leqslant a_0(t)+a_1(t)|x|^{\sigma}$$

H2　存在非负函数 $R\in C[0,1]$,使得对任意的 $t\in J$ 及任意的 $x,y\in\mathbb{R}$,有

$$|f(t,x)-f(t,y)|\leqslant R(t)|x-y|$$

记 $\|a_0\|_\infty=\max\limits_{0\leqslant t\leqslant 1}|a_0(t)|$, $\|a_1\|_\infty=\max\limits_{0\leqslant t\leqslant 1}|a_1(t)|$, $\|R\|_\infty=\max\limits_{0\leqslant t\leqslant 1}|R(t)|$.

定理 1　假设 H0,H1 成立,且 $0<\sigma<1$,则边值问题(1)至少存在 1 个解.

证明　取 $r_1\geqslant\max\{M_h,2(N_0\|a_0\|_\infty+N_2),(2N_0\|a_1\|_\infty)^{\frac{1}{1-\sigma}},2(N_1\|a_0\|_\infty+N_2),(2N_1\|a_1\|_\infty)^{\frac{1}{1-\sigma}}\}$.

令 $D=\{u\in PC:\|u\|_{PC}\leqslant r_1\}$,则 D 为 $PC(J,\mathbb{R})$ 中非空有界闭凸集.

对任意的 $u\in D$,当 $t\in[0,t_1]$ 时,结合引理 3 的 b、引理 4 以及 $1<\alpha<2$,可得

$$|Tu(t)|\leqslant|Au(t)|+|G(t)|\leqslant$$

$$\int_0^1\left[\frac{1}{\Gamma(\alpha)}+\frac{|\omega(s)|}{|\rho|}+\Big(\chi(s_m,s)\frac{(s_m-s)^{\alpha-2}}{|\rho|\Gamma(\alpha-1)}+\sum_{k=1}^{m}\frac{\chi(s_k,s)\int_{s_k}^{t_{k+1}}|g(\tau)|d\tau}{|\rho|\Gamma(\alpha-1)}(s_k-s)^{\alpha-2}\Big)\right]\cdot$$

$$|f(s,u(s))|ds+N_2\leqslant$$

$$(\|a_0\|_\infty+\|a_1\|_\infty r_1^\sigma)N_0+N_2\leqslant$$

$$N_0\|a_0\|_\infty+N_2+r_1^\sigma N_0\|a_1\|_\infty\leqslant$$

$$\frac{r_1}{2}+r_1^\sigma\frac{1}{2}r_1^{1-\sigma}=r_1$$

当 $t\in(t_k,s_k]$, $k=1,2,\cdots,m$ 时,$|Tu(t)|=|h_k(t)|\leqslant M_h\leqslant r_1$.

当 $t\in(s_k,t_{k+1}]$, $k=1,2,\cdots,m$ 时,

$$|Tu(t)|\leqslant|Au(t)|+|G(t)|\leqslant$$

$$\int_0^1\Big(\frac{1+s_m^{\alpha-1}}{\Gamma(\alpha)}+\frac{\chi(s_k,s)(s_k-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big)\cdot$$

$$|f(s,u(s))|ds+N_2\leqslant$$

$$(\|a_0\|_\infty+\|a_1\|_\infty r_1^\sigma)\frac{1+2s_m^{\alpha-1}}{\Gamma(\alpha)}+N_2\leqslant$$

$$N_1\|a_0\|_\infty+N_2+r_1^\sigma N_1\|a_1\|_\infty\leqslant r_1$$

因此,$\|T\|_{PC}\leqslant r_1$,故 $T(D)\subset D$. 由引理 5 可知 T 全连续,故由 Schauder 不动点定理可知 T 在 D 中至少存在 1 个不动点,即边值问题(1)在 $PC(J,\mathbb{R})$ 中至少存在 1 个解.

定理 2　假设 H0,H1 成立,如果 $\sigma=1$,且 $0<N_i\|a_1\|_\infty<1$, $i=0,1$,则边值问题(1)至少存在 1 个解.

证明　取 $r_2\geqslant\max\{M_h,\frac{N_0\|a_0\|_\infty+N_2}{1-N_0\|a_1\|_\infty},\frac{N_1\|a_0\|_\infty+N_2}{1-N_1\|a_1\|_\infty}\}$,令 $D=\{u\in PC:\|u\|_{PC}\leqslant r_2\}$,则 D 为 $PC(J,\mathbb{R})$ 中非空有界闭凸集,与定理 1 的证明类似,可得对任意的 $u\in D$,当 $t\in[0,t_1]$ 时,

$$|Tu(t)|\leqslant|Au(t)|+|G(t)|\leqslant$$

$$(\|a_0\|_\infty+\|a_1\|_\infty r_2)N_0+N_2\leqslant$$

$$(1-N_0\|a_1\|_\infty)r_2+r_2=r_2$$

当 $t\in(t_k,s_k]$, $k=1,2,\cdots,m$ 时,$|Tu(t)|=|h_k(t)|\leqslant M_h\leqslant r_2$.

当 $t\in(s_k,t_{k+1}]$, $k=1,2,\cdots,m$ 时,

$$|Tu(t)|\leqslant|Au(t)|+|G(t)|\leqslant$$

$$\frac{(1+2s_m^{\alpha-1})(\|a_0\|_\infty+\|a_1\|_\infty r_2)}{\Gamma(\alpha)}+N_2\leqslant$$

$$N_1\|a_0\|_\infty+N_2+r_2N_1\|a_1\|_\infty\leqslant r_2$$

因此，$\| T \|_{PC} \leqslant r_2$，故 $T(D) \subset D$. 由引理5可知 T 全连续，故由 Schauder 不动点定理可知 T 在 D 中至少存在1个不动点，即边值问题(1)在 $PC(J,\mathbb{R})$ 中至少存在1个解.

定理3 假设 H0,H2 成立，如果 $0 < N_i \| R \|_\infty < 1, i = 0,1$，则边值问题(1)存在唯一解.

证明 对任意的 $u_1, u_2 \in PC(J,\mathbb{R})$，当 $t \in [0, t_1]$ 时，

$$|Tu_2(t) - Tu_1(t)| \leqslant \| R \|_\infty \| u_2 - u_1 \|_{PC} \cdot \left(\frac{|\rho| + 1 + s_m^\alpha + 2\sum_{k=0}^{m} t_{k+1}^\alpha \| g \|_{L^1}}{|\rho| \Gamma(\alpha + 1)} + \frac{(1 - s_m) s_m^{\alpha-1} + s_m^{\alpha-1} \| g \|_{L^1}}{|\rho| \Gamma(\alpha)} \right) \leqslant N_0 \| R \|_\infty \| u_2 - u_1 \|_{PC}$$

当 $t \in (t_k, s_k], k = 1,2,\cdots,m$ 时，$|Tu_2(t) - Tu_1(t)| = 0$.

当 $t \in (s_k, t_{k+1}], k = 1,2,\cdots,m$ 时，

$$|Tu_2(t) - Tu_1(t)| \leqslant \frac{1 + 2s_m^{\alpha-1}}{\Gamma(\alpha)} \| R \|_\infty \| u_2 - u_1 \|_{PC} \leqslant N_1 \| R \|_\infty \| u_2 - u_1 \|_{PC}$$

因为，$0 < N_i \| \mathbb{R} \|_\infty < 1, i = 0,1$，故 $T: PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ 是压缩映射，由压缩映射原理可知 T 在 $PC(J,\mathbb{R})$ 上存在1个不动点，则边值问题(1)在 $PC(J,\mathbb{R})$ 中存在唯一解.

4 例 子

考虑边值问题

$$\begin{cases} {}^c\mathrm{D}_{0^+}^{\frac{3}{2}} u(t) = \dfrac{1}{(t+6)^2} + \dfrac{|u(t)|^{\frac{1}{2}}}{2+t}, \\ \quad t \in \left[0, \dfrac{1}{3}\right] \cup \left(\dfrac{2}{3}, 1\right] \\ u(t) = h_1(t) = \mathrm{e}^{-t}, t \in \left(\dfrac{1}{3}, \dfrac{2}{3}\right] \\ u'\left(\dfrac{2}{3}\right) = h_1'\left(\dfrac{2}{3}\right) = -\mathrm{e}^{-\frac{2}{3}} \\ u(0) = 0,\ u(1) = \int_0^1 \mathrm{e}^{-s} u(s) \mathrm{d}s \end{cases} \tag{13}$$

其中，$\alpha = \dfrac{3}{2}, m = 1, f(t,u) = \dfrac{1}{(t+6)^2} + \dfrac{|u|^{\frac{1}{2}}}{2+t}$，$h_1(t) = \mathrm{e}^{-t}, g(t) = \mathrm{e}^{-t}, u'(s_1) = h_1'\left(\dfrac{2}{3}\right) = -\mathrm{e}^{-\frac{2}{3}} = -0.5134$，易得 $\| g \|_{L^1} = \int_0^1 \mathrm{e}^{-s} \mathrm{d}s = 0.63212, \rho = \int_0^{\frac{1}{3}} s\mathrm{e}^{-s} \mathrm{d}s = 0.0446 \neq 0$，即 H0 成立，并且当 $t \in [0,1]$ 时，存在非负函数 $a_0(t) = \dfrac{1}{(t+6)^2} \in C[0,1], a_1(t) = \dfrac{1}{2+t} \in C[0,1]$，存在常数 $0 < \sigma = \dfrac{1}{2} < 1$，使得对任意的 $t \in [0,1]$ 及任意的 $x \in \mathbb{R}$，有 $|f(t,x)| = \dfrac{1}{(t+6)^2} + \dfrac{|x|^{\frac{1}{2}}}{2+t} = a_0(t) + a_1(t)|x|^{\frac{1}{2}}$，即 H1 成立.

综上可知定理1的所有条件均满足，根据定理1可知边值问题(13)至少存在1个解.

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