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多阶分数阶时滞微分方程的谱延迟校正法

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摘要：分数阶时滞微分方程(FDDEs)在物理、生物等众多领域有着广泛应用。针对分数阶时滞微分方程(FDDEs)，创造性地提出并应用谱延迟校正法(SDC)作为解决方案，构建一种基于双网格的 Legendre 延迟校正谱方法。引入双网格技术，对时间和空间离散进行优化处理，同时结合 Legendre 多项式进行谱延迟校正，大幅提升求解精度。制定预测步骤和校正步骤进行详尽误差分析。预测步骤以初步逼近的方式为解提供初始估计，通过校正步骤进一步细化解的近似，从而显著提高整体数值精度。数值实验结果表明，双网格 Legendre 延迟校正谱方法在处理分数阶时滞微分方程时成效卓著，极大地提高了精度，充分验证了理论结果的正确性。

关键词：多阶分数阶时滞微分方程；双网格谱延迟校正法；误差分析
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Spectral delay correction method for multi-order fractional delay differential equations

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Abstract: Fractional delay differential equations (FDDEs) have extensive applications in diverse fields such as physics and biology. The spectral delay correction (SDC) method for FDDEs was proposed as a solution, and a double-grid-based Legendre delay correction spectral method was developed. The double-grid technique was introduced to optimize the time and space discretization, while the Legendre polynomial was used for spectral delay correction, significantly enhancing the solution accuracy. A detailed error analysis was conducted through prediction and correction steps. The prediction step provided an initial estimate for the solution through an approximation, while the correction step further refined the approximation, thereby substantially improving the overall numerical accuracy. Numerical

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experiments demonstrate that the double-grid Legendre delay correction spectral method performed exceptionally well in solving FDDEs, greatly improving accuracy and fully verifying the correctness of the theoretical results.

Keywords: *mult-order fractional delay differential equations*; *double-grid spectral delay correction method*; *error analysis*

近年来，分数阶时滞微分方程（FDDEs）在流体流动、力学、化学、工程以及生物科学等多个领域得到了广泛应用，用以描述各种复杂现象。这类方程的研究吸引了大量科研工作者的关注[1-2]。由于分数阶偏微分方程通常难以获得精确解，解决这些问题仍然具有相当大的挑战性。尽管如此，已有一些有效的数值方法用于处理此类问题。例如，Rahimkhani 等[3] 提出了基于广义分数阶 Bernoulli 小波的数值方法；Shi 等[4] 构造了一类用于求解分数阶微分方程的运算矩阵方法；Wang 等[5] 开发了一种基于变换的切比雪夫多项式的数值技术，用于处理具有变系数的广义 FPDE，以描述复杂现象，这也进一步引起了科研工作者的关注[1-2]。最近，Guo 等[6] 将切比雪夫排序方法成功应用于 FDDEs。然而，关于分数阶微分方程（FDE）的谱延迟校正方法（SDC）的研究仍然较少。Lin 等[7] 提出了针对分数阶微分方程的 SDC 方法，并在文献 [8] 中给出了$2-\alpha$阶有限差分法的预测校正迭代的收敛速度推导，针对部分非均匀网格进行分析。然而，关于整个 SDC 方案的收敛速度，尚缺乏严格的理论分析。

1　预备知识

本文的目的是研究非线性分数阶时滞微分方程（FDDEs）的数值解：

$$\begin{cases} D^{\alpha}u(t)-D^{\alpha_1}u(t)-D^{\alpha_1}u(qt)=g(t,u(t),u(qt)), \\ t\in(0,T) \\ u(0)=0, u'(0)=0, \quad t\in(0,T) \end{cases} \tag{1}$$

其中，$\alpha\in(1,2),\alpha_1\in(0,1),q\in(0,1),f\in C(0,T)$，假设满足下列 Lipschitz 条件：

$$|f(s,y_1,y)-f(s,y_2,y)|\leqslant L|y_1-y_2|$$
$$|f(s,y,y_1)-f(s,y,y_2)|\leqslant L|y_1-y_2|$$

令D^{α}为α阶的 Caputo 导数，定义为

$$D^{\alpha}u=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}u^{(n)}(\tau)\mathrm{d}\tau, \quad t>0 \tag{2}$$

在本文中，将对式 (1) 应用双网格显式 SDC 方法，通过与文献 [9] 中类似的论点，可以将方程转换为以下等价的 VolterraFredholm 积分方程：

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s),u(qs))\mathrm{d}s+ \frac{1}{\Gamma(\alpha-\alpha_1)}\int_0^t(t-s)^{\alpha-\alpha_1-1}(u(s)+u(qs))\mathrm{d}s \tag{3}$$

1.1　全局划分

首先用节点对区间I_n进行粗划分：

$$\xi_0=0,\ \xi_{N^c}=T，并且\ \xi_{\mu-1}:=q\xi_{\mu}\,(2\leqslant\mu\leqslant N^c)$$

式中，N^c为正整数。显然对于任意$\Lambda_\mu=[\xi_{\mu-1},\xi_\mu]$，都有$q(\Lambda_2)\subseteq\Lambda_1$，$\Lambda_{\mu-1}=q\Lambda_\mu$，并且，$3\leqslant\mu\leqslant N^c$。每个子区间$[\xi_{\mu-1},\xi_\mu]$的细化分定义如下：

$$I_h^{(N^c)}:=\left\{t_j^{(\mu)}:\xi_{N^c-1}=t_0^{(N^c)}<t_1^{(N^c)}<\cdots<t_{N_\mu^f}^{(N^c)}=\xi_{N^c}\right\},$$
$$I_h^{(\mu-1)}:=qI_h^{(\mu)}$$

为了更好地说明网格划分，这里举一个简单的例子：

$$I_h^{(\mu-1)}:=\left\{\xi_{\mu-2},t_1^{(\mu-1)},t_2^{(\mu-1)},\xi_{\mu-1}\right\}$$
$$I_h^{(\mu)}:=\left\{\xi_{\mu-1},t_1^{(\mu)},t_2^{(\mu)},\xi_{\mu}\right\}$$

并且

$$\xi_{\mu-2}=q\xi_{\mu-1},\quad \xi_{\mu-1}=q\xi_{\mu},$$
$$t_1^{(\mu-1)}=qt_1^{(\mu)},\quad t_2^{(\mu-1)}=qt_2^{(\mu)}$$

方便起见，这里将先前的全局网格重命名为

$$I_h:=\{t_n:0=t_0<t_1<\cdots<t_N=T\}$$

并且定义$h_n=t_n-t_{n-1}$，$I_n=(t_{n-1},t_n]$。

1.2　区间I_n上的局部划分

对区间I_n进行划分，对于任意整数$M\geqslant0$，用$\{x_{n,j},\omega_{n,j}\}_{j=0}^M$表示区间$(-1,1)$上标准 Legendre-Gauss 插值的节点和对应的 Christoffel 数。设$\mathcal{P}_M(I_n)$是区间I_n上M次多项式的集合，并且$\{t_{n,j}\}_{j=0}^M$是区间I_n上转换的 Legendre-Gauss 正交节点：

$$t_{n,j}=\frac{1}{2}\left(h_nx_{n,j}+t_{n-1}+t_n\right),\quad 1\leqslant n\leqslant N,\quad 0\leqslant j\leqslant M \tag{4}$$

根据标准 Legendre-Gauss 正交的性质，可以得出这样的结论：

$$\int_{I_n}\phi(t)\mathrm{d}t=\frac{h_n}{2}\sum_{j=0}^{M}\phi(t_{n,j})\omega_j,\quad \forall\phi\in\mathcal{P}_{2M+1}(I_n) \tag{5}$$

此外，用$\mathcal{I}_{s,M}:C(t_{n-1},t_n)\to\mathcal{P}_M(t_{n-1},t_n)$表示$s$方向上转换的 Legendre-Gauss 插值算子，使得

$$\mathcal{I}^n_{s,M}v(t_{n,j})=v(t_{n,j}),\quad 0\leqslant j\leqslant M \tag{6}$$

设$\mathcal{L}^n_s$为定义在$(t_{n-1},t_n]$上的分段插值算子，有

$$\begin{cases}\mathcal{L}^n_s v(s)=v(t_0), & \forall s\in(t_0,t_{1,0}]\\ \mathcal{L}^n_s v(s)=v(t_{n-1,M}), & \forall s\in(t_{n-1},t_{n,0}]\\ \mathcal{L}^n_s v(s)=v(t_{n,j-1}), & \forall s\in(t_{n,j-1},t_{n,j}]\\ \mathcal{L}^n_s v(s)=v(t_{n,M}), & \forall s\in(t_{n,M},t_n]\end{cases} \tag{7}$$

式中，$1\leqslant j\leqslant M$。

2 显式双网格谱延迟矫正法

本节详细介绍双网格谱延迟矫正法，方程 (1) 转换为以下等价的 Volterra 积分方程[8]：

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^{t_{n-1}}(t-s)^{\alpha-1}f(s,u(s),u(qs))\mathrm{d}s+\frac{1}{\Gamma(\alpha-\alpha_1)}\int_0^{t_{n-1}}(t-s)^{\alpha-\alpha_1-1}(u(s)+u(qs))\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_{t_{n-1}}^{t}(t-s)^{\alpha-1}f(s,u(s),u(qs))\mathrm{d}s+\frac{1}{\Gamma(\alpha-\alpha_1)}\int_{t_{n-1}}^{t}(t-s)^{\alpha-\alpha_1-1}(u(s)+u(qs))\mathrm{d}s \tag{8}$$

将第n个解表示为$y_n(t)$，即

$$u^n(t)=y(t),\quad \forall t\in I_n,\quad 1\leqslant n\leqslant N \tag{9}$$

对于任意$t\in I_n$，式 (3) 等价于式 (8)。

接着考虑区间$I_{\hat{k}}=[qt_{k-1},qt_k]$，其中，$k>1$。根据网格划分的特征，存在唯一区间$I_j$，且$1\leqslant j<k$，使得$I_{\hat{k}}\subseteq I_j$。相应地表示：

$$u^{\hat{k}}(t)=u^j(t),\quad t\in I_{\hat{k}}$$

因此可得

$$u^n(t)=\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{n-1}\int_{I_k}(t-s)^{\alpha-1}f(s,u(s),u(qs))\mathrm{d}s+\frac{1}{\Gamma(\alpha-\alpha_1)}\sum_{k=1}^{n-1}\int_{I_k}(t-s)^{\alpha-\alpha_1-1}(u(s)+u(qs))\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_{t_n}^{t}(t-\xi)^{\alpha-1}f(\xi,u(\xi),u(q\xi))\mathrm{d}s+\frac{1}{\Gamma(\alpha-\alpha_1)}\int_{t_n}^{t}(t-\xi)^{\alpha-\alpha_1-1}(u(\xi)+u(q\xi))\mathrm{d}s \tag{10}$$

下文分两个步骤构造式 (10) 的显式 SDC 方法，包括预测步骤和矫正步骤。

2.1 预测步骤

取$U^n(t)\in\mathcal{P}_M(I_n)$，使得对于$1\leqslant j\leqslant M$，都有

$$U^n(t_{n,j})=\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{n-1}\int_{I_k}(t_{n,j}-s)^{\alpha-1}\mathcal{I}^k_{s,M}f\left(s,U^k(s),U^{\hat{k}}(qs)\right)\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{n-1}\int_{t_{n-1}}^{t_{n,j}}(t_{n,j}-s)^{\alpha-1}\mathcal{L}^n_\xi f\left(\xi,U^n(\xi),\mathcal{L}^n_{q\xi}U^{\hat{n}}(q\xi)\right)\mathrm{d}\xi+\frac{1}{\Gamma(\alpha-\alpha_1)}\sum_{k=1}^{n-1}\int_{I_k}(t_{n,j}-s)^{\alpha-\alpha_1-1}\mathcal{I}^k_{s,M}\left(U^k(s)+U^{\hat{k}}(qs)\right)\mathrm{d}s+\frac{1}{\Gamma(\alpha-\alpha_1)}\int_{t_{n-1}}^{t_{n,j}}(t_{n,j}-s)^{\alpha-\alpha_1-1}\mathcal{L}^n_\xi\left(U^n(\xi)+U^{\hat{n}}(q\xi)\right)\mathrm{d}\xi \tag{11}$$

式中：$U^k(t)\in\mathcal{P}_M(I_k)$是区间$I_k$上的数值解；$U^{\hat{k}}(t)\in\mathcal{P}_M\left(I_{\hat{k}}\right)$是延迟区间$I_{\hat{k}}$上的数值解。特别地，当$n>1$时，对于$\forall\xi\in(t_{n-1},t_{n,0}]$，有

$$\mathcal{L}^n_\xi f\left(\xi,U^n(\xi),U^{\hat{n}}(q\xi)\right)=f\left(t_{n-1},U^{n-1}(t_{n-1,M}),U^{\widehat{n-1}}(qt_{n-1,M})\right)$$

$$\mathcal{L}^n_\xi\left(U^n(\xi)+U^{\hat{n}}(q\xi)\right)=U^{n-1}(t_{n-1,M})+U^{\widehat{n-1}}(qt_{n-1,M})$$

2.2 矫正步骤

通过预测步骤得到了数值解$U^n(t)$，现在对其作出矫正。令$U^n_{[l]}(t)$为式 (11) 的$l-k$步校正步骤的数值解。$U^n_{[l]}(t)$的残差函数定义如下：

$$\delta^n_{[l]}(t)=U^n_{[l]}(t)-\frac{1}{\Gamma(\alpha)}\left(\sum_{k=1}^{n-1}\int_{I_k}(t-s)^{\alpha-1}f\left(s,U^k_{[l]}(s),U^{\hat{k}}_{[l]}(qs)\right)\mathrm{d}s+\int_{t_{n-1}}^{t}(t-\xi)^{\alpha-1}f\left(\xi,U^n_{[l]}(\xi),U^{\hat{n}}_{[l]}(q\xi)\right)\mathrm{d}\xi\right)-\frac{1}{\Gamma(\alpha-\alpha_1)}\left(\sum_{k=1}^{n-1}\int_{I_k}(t-s)^{\alpha-\alpha_1-1}\left(U^k_{[l]}(s)+U^{\hat{k}}_{[l]}(qs)\right)\mathrm{d}s+\int_{t_{n-1}}^{t}(t-\xi)^{\alpha-\alpha_1-1}\left(U^k_{[l]}(\xi)+U^{\hat{k}}_{[l]}(q\xi)\right)\mathrm{d}\xi\right) \tag{12}$$

由于式 (12) 中的积分项无法精确计算，因此近似用$\delta^{[l]}_{n,j}$表示$\delta^n_{[l]}(t_{n,j})$。

$$\delta_{n,j}^{[l]} = U_{[l]}^{n}\left(t_{n,j}\right) - \frac{1}{\Gamma(\alpha)}\left(\sum_{k=1}^{n-1}\int_{I_k}(t_{n,j}-s)^{\alpha-1}\mathcal{I}_{s,M}^{k} f\left(s,U_{[l]}^{k}(s),U_{[l]}^{\hat{k}}(qs)\right)\mathrm{d}s + \int_{t_{n-1}}^{t_{n,j}}(t_{n,j}-\xi)^{\alpha-1}\mathcal{I}_{s,M}^{n} f\left(\xi,U_{[l]}^{n}(\xi),U_{[l]}^{\hat{n}}(q\xi)\right)\mathrm{d}\xi\right) + \frac{1}{\Gamma(\alpha-\alpha_1)}\left(\sum_{k=1}^{n-1}\int_{I_k}(t_{n,j}-s)^{\alpha-\alpha_1-1}\mathcal{I}_{s,M}^{k}\left(U_{[l]}^{k}(s)+U_{[l]}^{\hat{k}}(qs)\right)\mathrm{d}s + \int_{t_{n-1}}^{t_{n,j}}(t_{n,j}-\xi)^{\alpha-\alpha_1-1}\mathcal{I}_{s,M}^{n}\left(U_{[l]}^{n}(s)+U_{[l]}^{\hat{n}}(qs)\right)\mathrm{d}\xi\right) \tag{13}$$

接下来，通过$e_{[l]}^{n}(t) = u^{n}(t) - U_{[l]}^{n}(t)$来定义第一个校正步骤的误差函数。从式 (11) 中很容易推断出：

$$e_{[l]}^{n}(t) + U_{[l]}^{n}(t) = \frac{1}{\Gamma(\alpha)}\left(\sum_{k=1}^{n-1}\int_{I_k}(t-s)^{\alpha-1} f\left(s, e_{[l]}^{k}(s)+U_{[l]}^{k}(s), e_{[l]}^{\hat{k}}(qs)+U_{[l]}^{\hat{k}}(qs)\right)\mathrm{d}s + \int_{t_{n-1}}^{t}(t-\xi)^{\alpha-1} f\left(\xi, e_{[l]}^{n}(\xi)+U_{[l]}^{n}(\xi), e_{[l]}^{\hat{n}}(q\xi)+U_{[l]}^{\hat{n}}(q\xi)\right)\mathrm{d}\xi\right) + \frac{1}{\Gamma(\alpha-\alpha_1)}\left(\sum_{k=1}^{n-1}\int_{I_k}(t-s)^{\alpha-\alpha_1-1}\left(e_{[l]}^{k}(s)+U^{k}{}_{[l]}(s)+e_{[l]}^{\hat{k}}(qs)+U_{[l]}^{\hat{k}}(qs)\right)\mathrm{d}s + \int_{t_{n-1}}^{t_{n,j}}(t-\xi)^{\alpha-\alpha_1-1}\left(e_{[l]}^{n}(\xi)+U^{n}{}_{[l]}(\xi)+e_{[l]}^{\hat{n}}(q\xi)+U_{[l]}^{\hat{n}}(q\xi)\right)\mathrm{d}\xi\right) \tag{14}$$

使用以下显格式来近似$e_{[l]}^{n}(t)$的解：对于$E_{[l]}^{n}(t)\in\mathcal{P}_M(I_n)$，使得：

$$E_{[l]}^{n}\left(t_{n,j}\right) = \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{n-1}\int_{I_k}(t_{n,j}-s)^{\alpha-1}\mathcal{I}_{s,M}^{k}\left(f\left(s,E_{[l]}^{k}(s)+U_{[l]}^{k}(s),E_{[l]}^{\hat{k}}(qs)+U_{[l]}^{\hat{k}}(qs)\right) - f\left(s,U_{[l]}^{k}(s),U_{[l]}^{\hat{k}}(qs)\right)\mathrm{d}s\right) + \int_{t_{n-1}}^{t_{n,j}}(t_{n,j}-\xi)^{\alpha-1}\mathcal{L}_{\xi}^{n}\left(f\left(\xi,E_{[l]}^{n}(\xi)+U_{[l]}^{n}(\xi),E_{[l]}^{\hat{n}}(q\xi)+\mathcal{L}_{q\xi}^{\hat{n}}U_{[l]}^{\hat{n}}(q\xi)\right) - f\left(\xi,U_{[l]}^{n}(\xi),\mathcal{L}_{q\xi}^{\hat{n}}U_{[l]}^{\hat{n}}(q\xi)\right)\mathrm{d}\xi\right) + \frac{1}{\Gamma(\alpha-\alpha_1)}\left(\sum_{k=1}^{n-1}\int_{I_k}(t-s)^{\alpha-\alpha_1-1}\left(E_{[l]}^{k}(s)+E_{[l]}^{\hat{k}}(qs)\right)\mathrm{d}s\right) + \int_{t_{n-1}}^{t}(t-\xi)^{\alpha-\alpha_1-1}\left(E_{[l]}^{k}(\xi)+E_{[l]}^{\hat{k}}(q\xi)\right)\mathrm{d}\xi - \delta_{n,j}^{[l]},\quad 1\leqslant j\leqslant M \tag{15}$$

那么，校正后的数值解如下

$$U_{[l+1]}^{n}(t_{n,j}) = E_{[l]}^{n}(t_{n,j}) + U_{[l]}^{n}(t_{n,j}) \tag{16}$$

为了更好地说明矫正步骤，下面给出了显式双网格谱延迟矫正步骤的算法：首先，计算出当$l<L$时，对$n=1,2,\cdots,N$，$\left\{U^{n}(t_{n,j})\right\}_{j=0}^{M}$的值。当$l=0$时，计算$\left\{U_{[l]}^{n}(t_{n,j})\right\}_{j=0}^{M} = \left\{U^{n}(t_{n,j})\right\}_{j=0}^{M}$。通过式 (13) 计算出$\left\{\delta_{n,j}^{[l]}\right\}_{j=0}^{M}$，通过式 (15) 计算出$\left\{E_{[l]}^{n}\left(t_{n,j}\right)\right\}_{j=0}^{M}$，最后利用式 (16) 更新$\left\{U^{n}(t_{n,j})\right\}_{j=0}^{M}$。

为了确保 SDC 方法的有效性，一个关键问题是如何精确计算式 (15) 中的积分。根据文献 [2] 所述，由于积分核中存在弱奇异性，需要设计两个加权插值正交公式：对于任意的$\phi(s)\in\mathcal{P}_M(I_k)$与$\varphi(s)\in\mathcal{P}_M(I_n)$，令

$$\int_{I_k}(t-s)^{\alpha-1}\phi(s)\mathrm{d}s = \sum_{j=0}^{M_k}\phi(t_{k,j})\tilde{\omega}_{k,j}(t),\quad t\in I_k,\quad 1\leqslant k<n$$

$$\int_{t_{n-1}}^{t}(t-s)^{\alpha-1}\varphi(s)\mathrm{d}s = \sum_{j=0}^{M_n}\varphi(t_{n,j})\hat{\omega}_{n,j}(t),\quad t\in I_n$$

其中，

$$\tilde{\omega}_{k,j}(t) = \int_{I_k}(t-s)^{\alpha-1}l_{k,j}(s)\mathrm{d}s$$

$$\hat{\omega}_{n,j}(t) = \int_{t_{n-1}}^{t}(t-s)^{\alpha-1}l_{n,j}(s)\mathrm{d}s$$

$\left\{l_{k,j}(s)\right\}_{j=0}^{M}$是对应于配置点$\left\{t_{k,j}\right\}_{j=0}^{M}$的拉格朗日基本多项式。根据勒让德多项式的性质，文献 [10] 中的类似论点给出了

$$\tilde{\omega}_{k,j}(t) = \sum_{m=0}^{M_k}c_{k,m}^{j}\sum_{p=0}^{m}a_m^{p}\left(\sum_{l=0}^{p}b_{k,l}^{p}(t-t_{k-1})^{l+\alpha} - b_{k,p}^{p}(t-t_k)^{p+\alpha}\right)$$

$$\hat{\omega}_{n,j}(t) = \sum_{m=0}^{M_n}c_{n,m}^{j}\sum_{p=0}^{m}a_m^{p}\sum_{l=0}^{p}b_{n,l}^{p}(t-t_{n-1})^{l+\alpha}$$

其中，

$$a_m^{p} = \frac{\Gamma(m+p+1)}{\Gamma^2(p+1)\Gamma(m+1-p)}$$

$$b_{k,l}^{p} = \frac{(-1)^{p-l}\Gamma(\alpha)}{\Gamma(\alpha+l+1)}\frac{\Gamma(p+1)}{\Gamma(p-l+1)}h_k^{-l}$$

$$c_{k,m}^{j} = \frac{2m+1}{2}L_{k,m}(t_{k,j})\omega_{k,j}$$

这里，$L_{k,m}(t) = L_m\left(\dfrac{2t-t_{k-1}-t_k}{h_n}\right)$，并且$L_m(t)$是$(-1,1)$上的多项式。使用以上两个求积公式，可以

精确地计算式 (15) 的中的积分 $\int_{I_k}(t_{n,j}-s)^{\alpha-1}\mathcal{I}^k_{s,M}\cdot f\left(s,U^k_{[l]}(s),U^k_{[l]}(qs)\right)\mathrm{d}s$和$\int_{t_{n-1}}^{t_{n,j}}(t_{n,j}-\xi)^{\alpha-1}\mathcal{I}^n_{\xi,M}f\left(\xi,U^n_{[l]}(\xi),U^n_{[l]}(q\xi)\right)\mathrm{d}\xi$。

3 误差分析

本节分析式 (11) 的数值误差。假设式 (1) 中的函数f总是满足以下 Lipschitz 条件：

$$\begin{cases}|f(s,y_1,y)-f(s,y_2,y)|\leqslant L|y_1-y_2|\\|f(s,y,y_1)-f(s,y,y_2)|\leqslant L|y_1-y_2|\end{cases}\tag{17}$$

引理 1[9] 对于任意$v\in H^m(I_n)$，且$1\leqslant m\leqslant M+1$，$m\in Z$，都有

$$\left\|v-\mathcal{I}_{t,M}v\right\|_{L^2(I_n)}\leqslant ch_n^mM_n^{-m}\left\|\partial_t^mv\right\|_{L^2(I_n)}\tag{18}$$

其中，$H^m(I_n)$是常规 Soblev 空间。

引理 2[11] 设$\{k_j\}$和$\{\rho_j\}(j\geqslant 0)$是给定的非负序列，序列$\{\varepsilon_n\}$满足$\varepsilon_0\leqslant\rho_0$，则

$$\varepsilon_n\leqslant\rho_n+\sum_{j=0}^{n-1}q_j+\sum_{j=0}^{n-1}k_j\varepsilon_j,\quad n\geqslant 1\tag{19}$$

其中，$q_j\geqslant 0(j\geqslant 0)$，那么

$$\varepsilon_n\leqslant\rho_n+\sum_{j=0}^{n-1}(q_j+k_j\rho_j)\exp\left(\sum_{j=0}^{n-1}k_j\right),\quad n\geqslant 1\tag{20}$$

3.1 双网格 SDC 方法在预测步骤中的误差分析

对式 (11) 的数值误差进行分析，联立式 (10) 和式 (11) 可以得到

$$u^n(t_{n,j})-U^n(t_{n,j}t)=\sum_{i=1}^4B_i(t_{n,j})\tag{21}$$

其中，

$$B_1\left(t_{n,j}\right)=\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{n-1}\int_{I_k}\left(t_{n,j}-s\right)^{\alpha-1}\left(f(s,u^k(s),u^k(qs))-\mathcal{I}^k_{s,M}f(s,U^k(s),U^{\hat{k}}(qs))\right)\mathrm{d}s$$

$$B_2\left(t_{n,j}\right)=\frac{1}{\Gamma(\alpha)}\int_{t_{n-1}}^{t_{n,j}}\left(t_{n,j}-\xi\right)^{\alpha-1}\left(f(\xi,u^n(\xi),u^n(q\xi))-\mathcal{L}^n_{\xi}f\left(\xi,U^n(\xi),U^{\hat{n}}(q\xi)\right)\right)\mathrm{d}\xi$$

$$B_3(t_{n,j})=\frac{1}{\Gamma(\alpha-\alpha_1)}\sum_{k=1}^{n-1}\int_{I_k}(t_{n,j}-s)^{\alpha-\alpha_1-1}\left(u^k(s)-U^k(s)+(u^{\hat{k}}(qs)-U^{\hat{k}}(qs)\right)\mathrm{d}s$$

$$B_4(t_{n,j})=\frac{1}{\Gamma(\alpha-\alpha_1)}\int_{t_{n-1}}^{t_{n,j}}(t_{n,j}-s)^{\alpha-\alpha_1-1}\left((u^n(\xi)+u^{\hat{n}}(\xi))-\mathcal{L}^n_{\xi}\left(U^n(\xi)+U^{\hat{n}}(q\xi)\right)\right)\mathrm{d}\xi$$

下面逐个估计$\left\{B_i\left(t_{n,j}\right)\right\}_{i=1}^4$。

引理 3 对于任意整数$1\leqslant m\leqslant M+1$，都有

$$\begin{aligned}|B_1(t_{n,j})|^2\leqslant{}&c\sum_{k=1}^{n-1}h_k\max_{0\leqslant j\leqslant M}\left|e^k_{[0]}(t_{k,j})\right|^2+\\&ch_1^{2m}M^{-2m}\left\|\partial_s^mu(s)\right\|^2_{L^2(I_1)}+\\&c\sum_{k=1}^{n-1}h_k^{2m}M^{-2m}\left\|\partial_s^mf(t,u(s),u(qs))\right\|^2_{L^2(I_k)}\end{aligned}\tag{22}$$

并且

$$\begin{aligned}\left|B_3\left(t_{n,j}\right)\right|^2\leqslant{}&c\sum_{k=1}^{n-1}h_k\max_{0\leqslant j\leqslant M}\left(\left|e^k_{[0]}(t_{k,j})\right|^2\right)+\\&c\sum_{k=1}^{n-1}h_k^{2m}M^{-2m}\left\|\partial_s^m(u(s)+u(qs))\right\|^2_{L^2(I_k)}+\\&ch_1^{2m}M^{-2m}\left\|\partial_s^m(u(s))\right\|^2_{L^2(I_1)}\end{aligned}\tag{23}$$

这里要求不等式的右侧在区间上是有界的。

证明 设$V(s)$是$[0,t_{n-1}]$上定义的全局函数，使得

$$\begin{aligned}V(s)|_{s\in I_k}:={}&f\left(s,u^k(s),u^{\hat{k}}(qs)\right)-\\&\mathcal{I}^k_{s,M}f\left(s,U^k(s),U^{\hat{k}}(qs)\right)\end{aligned}\tag{24}$$

于是，$B_1\left(t_{n,j}\right)$可以重写为

$$B_1\left(t_{n,j}\right)=\frac{1}{\Gamma(\alpha)}\int_0^{t_{n-1}}(t_{n,j}-s)^{\alpha-1}V(s)\mathrm{d}s\tag{25}$$

将 Cauchy-Schwarz 不等式应用到式 (25) 中，得到

$$\begin{aligned}\left|B_1\left(t_{n,j}\right)\right|^2\leqslant{}&\frac{1}{\Gamma^2(\alpha)}\int_0^{t_{n-1}}\left(t_{n,j}-s\right)^{2(\alpha-1)}\mathrm{d}s\int_0^{t_{n-1}}V^2(s)\mathrm{d}s\leqslant\\&\frac{T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\sum_{k=1}^{n-1}\int_{I_k}\left(f\left(s,u^k(s),u^{\hat{k}}(qs)\right)-\right.\\&\left.\mathcal{I}^k_{s,M}f\left(s,U^k(s),U^{\hat{k}}(qs)\right)\right)^2\mathrm{d}s\leqslant\\&\frac{2T^{2\alpha-1}}{(2\alpha-1)\Gamma^2(\alpha)}\left(B_{11}+B_{12}\right)\end{aligned}\tag{26}$$

其中，

$$B_{11}=\sum_{k=1}^{n-1}\int_{I_k}\left|(\mathcal{I}-\mathcal{I}^k_{s,M})f\left(s,u^k(s),u^{\hat{k}}(qs)\right)\right|^2\mathrm{d}s$$

$$\begin{aligned}B_{12}={}&\sum_{k=1}^{n-1}\int_{I_k}\left|\mathcal{I}^k_{s,M}f\left(s,u^k(s),u^{\hat{k}}(qs)\right)-\right.\\&\left.f\left(s,U^k(s),U^{\hat{k}}(qs)\right)\right|^2\mathrm{d}s\end{aligned}$$

这里$\mathcal{I}$是恒等算子。接着由式 (18) 推算出

$$B_{11} \leqslant c\sum_{k=1}^{n-1} h_k^{2m} M_k^{-2m} \left\|\partial_s^m f(s,u(s),u(qs))\right\|_{L^2(I_k)}^2 \tag{27}$$

此外，根据式 (5) 和式 (17)，可推算出

$$B_{12} \leqslant L^2\sum_{k=1}^{n-1}\frac{h_k}{2}\sum_{j=0}^{M}\Big(f\big(t_{k,j},u^k(t_{k,j}),u^{\hat{k}}(qt_{k,j})\big)-f\big(t_{k,j},U^k(t_{k,j}),U^{\hat{k}}(qt_{k,j})\big)\Big)^2\omega_j \leqslant$$
$$L^2\sum_{k=1}^{n-1}\frac{h_k}{2}\sum_{j=0}^{M}\Big(\big(u^k(t_{k,j})-U^k(t_{k,j})\big)^2+\big(u^k(qt_{k,j})-U^k(qt_{k,j})\big)^2\Big)\omega_j \tag{28}$$

显然

$$\big(u^k(t_{k,j})-U^k(t_{k,j})\big)^2 \leqslant \max_{0\leqslant j\leqslant M}\left|e_{[0]}^k(t_{k,j})\right|^2$$

此外，如果$\hat{k}>1$，那么$qt_{k,j}$是区间I_k上的勒让德高斯节点。因此有

$$h_k\big(u^{\hat{k}}(qt_{k,j})-U^{\hat{k}}(qt_{k,j})\big)^2 \leqslant \frac{1}{q}h_{\hat{k}}\max_{0\leqslant j\leqslant M}\left|e_{[0]}^{\hat{k}}(t_{\hat{k},j})\right|^2$$

此外，如果$\hat{k}=1$，那么有

$$\frac{h_k}{2}\sum_{j=0}^{M}\big(u^{\hat{k}}(qt_{k,j})-U^{\hat{k}}(qt_{k,j})\big)^2\omega_j=\int_{I_k}\big(\mathcal{I}_{s,M}^k u^{\hat{k}}(qs)-U^{\hat{k}}(qs)\big)^2\mathrm{d}s \leqslant$$
$$2\int_{I_k}\big(\mathcal{I}_{s,M}^k u^{\hat{k}}(qs)-u^{\hat{k}}(qs)\big)^2\mathrm{d}s+2\int_{I_k}\big(u^{\hat{k}}(qs)-U^{\hat{k}}(qs)\big)^2\mathrm{d}s \leqslant$$
$$ch_k^{2m}M^{-2m}\left\|\partial_s^m u(qs)\right\|_{L^2(I_k)}^2+\frac{2}{q}\int_{I_{\hat{k}}}\big(u^{\hat{k}}(t)-U^{\hat{k}}(t)\big)^2\mathrm{d}t \leqslant$$
$$ch_k^{2m}M^{-2m}\left\|\partial_s^m u(qs)\right\|_{L^2(I_k)}^2+\frac{4}{q}\int_{I_1}\big(u^1(t)-\mathcal{I}_{t,M}^1 u^1(t)\big)^2\mathrm{d}t+\frac{4}{q}\int_{I_1}\big(\mathcal{I}_{t,M}^1 u^1(t)-U^1(t)\big)^2\mathrm{d}t \leqslant$$
$$ch_k^{2m}M^{-2m}\left\|\partial_s^m u(qs)\right\|_{L^2(I_k)}^2+ch_1^{2m}M^{-2m}\left\|\partial_t^m u\right\|_{L^2(I_1)}^2+\frac{2h_1}{q}\sum_{j=0}^{M}(u^1(t_{1,j})-U^1(t_{1,j}))^2\omega_j \leqslant$$
$$\frac{c}{q}h_1^{2m}M^{-2m}\left\|\partial_s^m u(t)\right\|_{L^2(I_1)}^2+ch_1^{2m}M^{-2m}\left\|\partial_t^m u\right\|_{L^2(I_1)}^2+ch_1\max_{0\leqslant j\leqslant M}\left|e_{[0]}^1(t_{1,j})\right|^2 \leqslant$$
$$ch_1^{2m}M^{-2m}\left\|\partial_s^m u\right\|_{L^2(I_1)}^2+ch_1\max_{0\leqslant j\leqslant M}\left|e_{[0]}^1(t_{1,j})\right|^2$$

因此，

$$B_{12} \leqslant c\sum_{k=1}^{n-1}h_k\max_{0\leqslant j\leqslant M}\left|e_{[0]}^k(t_{k,j})\right|^2+ch_1^{2m}M^{-2m}\left\|\partial_s^m u\right\|_{L^2(I_1)}^2 \tag{29}$$

由此完成证明。

□

引理 4　如果对于整数$1\leqslant m\leqslant M+1$有$u^n\in H^m(I_n)$，那么

$$|B_2(t_{n,j})|^2 \leqslant ch_n^{2\alpha}\Big(\left\|f(s,u^n(s),u^{\hat{n}}(qs))\right\|_{L^\infty(I_n)}^2+\left\|u\right\|_{L^\infty(I_n)}^2\Big)+ch_n^{2\alpha}\max_{0\leqslant i\leqslant M}\left|e_{[0]}^n(t_{n,i})\right|^2+ch_n^{2\alpha-1}\max_{\substack{0\leqslant i\leqslant M\\1\leqslant k\leqslant n-1}}h_k\left|e_{[0]}^n(t_{n,i})\right|^2 \tag{30}$$

并且

$$|B_4(t_{n,j})|^2 \leqslant ch_n^{2(\alpha-\alpha_1)}\Big(\left\|u^n(s)+u^{\hat{n}}(qs)\right\|_{L^\infty(I_n)}^2+\left\|u\right\|_{L^\infty(I_n)}^2\Big)+ch_n^{2(\alpha-\alpha_1)}\max_{0\leqslant i\leqslant M}\left|e_{[0]}^n(t_{n,i})\right|^2+$$
$$ch_n^{2(\alpha-\alpha_1)-1}\max_{\substack{0\leqslant i\leqslant M\\0\leqslant k\leqslant m-1}}h_k\left|e_{[0]}^{\widehat{n}}(qt_{n,i})\right|^2 \tag{31}$$

证明：根据式 (5) 和式 (17)，可推出

$$|B_2(t_{n,j})|^2 = \left(\frac{1}{\Gamma(\alpha)}\int_{t_{n-1}}^{t_{n,j}}(t_{n,j}-\xi)^{\alpha-1}(f(\xi,u^n(\xi),u^{\hat{n}}(q\xi)) - \mathcal{L}_\xi^n f(\xi,U^n(\xi),\mathcal{L}_{q\xi}^{\hat{n}}U^{\hat{n}}(q\xi)))\mathrm{d}\xi\right)^2 \leqslant \frac{2}{\Gamma^2(\alpha)}(B_{21}(t_{n,j})+B_{22}(t_{n,j})) \tag{32}$$

其中

$$B_{21}(t) = \left(\int_{t_{n-1}}^{t}(t-\xi)^{\alpha-1}(f(\xi,u^n(\xi),u^{\hat{n}}(q\xi)) - \mathcal{L}_\xi^n f(\xi,u^n(\xi),\mathcal{L}_{q\xi}^{\hat{n}}u^{\hat{n}}(q\xi)))\mathrm{d}\xi\right)^2$$

$$B_{22}(t) = \left(\int_{t_{n-1}}^{t}(t-\xi)^{\alpha-1}(\mathcal{L}_\xi^n(f(\xi,u^n(\xi),u^{\hat{n}}(q\xi)) - f(\xi,U^n(\xi),\mathcal{L}_{q\xi}^{\hat{n}}U^{\hat{n}}(q\xi))))\mathrm{d}\xi\right)^2$$

接下来，通过 Cauchy-Schwarz 不等式和式 (7) 中$\mathcal{L}_\xi^n$的定义，可以得到

$$B_{21}\left(t_{n,j}\right) \leqslant \int_{t_{n-1}}^{t_{n,j}}(t_{n,j}-\xi)^{2\alpha-2}\mathrm{d}\xi\int_{t_{n-1}}^{t_{n,j}}(f(\xi,u^n(\xi),u^{\hat{n}}(q\xi)) - \mathcal{L}_\xi^n f(\xi,u^n(\xi),u^{\hat{n}}(q\xi)))^2\mathrm{d}\xi \leqslant ch_n^{2\alpha}\left\|f(s,u(s),u(qs))\right\|_{L^\infty(0,T)}^2 + \left\|u\right\|_{L^\infty(0,T)}^2 \tag{33}$$

同理可得

$$B_{22}\left(t_{n,j}\right) \leqslant \int_{t_{n-1}}^{t_{n,j}}(t_{n,j}-\xi)^{2\alpha-2}\mathrm{d}\xi\int_{t_{n-1}}^{t_{n,j}}(\mathcal{L}_\xi^n(f(\xi,u^n(\xi),u^{\hat{n}}(q\xi)) - f(\xi,U^n(\xi),U^{\hat{n}}(q\xi))))^2\mathrm{d}\xi \leqslant \frac{h_n^{2\alpha}}{2\alpha-1}\Big[\max_{0\leqslant i\leqslant M}|(f(t_{n,i},u^n(t_{n,i}),u^{\hat{n}}(qt_{n,i})) - f(t_{n,i},U^n(t_{n,i}),U^{\hat{n}}(qt_{n,i})))^2|+ f(t_{n-1,M},u^n(t_{n-1,M}),u^{\hat{n}}(qt_{n-1,M})) - f(t_{n-1,M},U^n(t_{n-1,M}),U^{\hat{n}}(qt_{n-1,M}))\Big] \leqslant h_n^{2\alpha}\Big[\max_{0\leqslant i\leqslant M}\left(\left|e_{[0]}^n(t_{n,i})\right|^2 + \left|e_{[0]}^{\hat{n}}(qt_{n,i})\right|^2\right) + \left|e_{[0]}^{n-1}(t_{n-1,M}) + e_0^{\widehat{n-1}}(qt_{n-1,M})\right|^2\Big] \tag{34}$$

由式 (31)～(33) 可以推导出式 (30) 的结果，由此完成了证明。

定理 1 对于任意的整数$1\leqslant m\leqslant M+1$，

$$\max_{0\leqslant j\leqslant M}\left|e_{[0]}^n(t_{n,j})\right|^2 \leqslant cT\exp(T)h_n^{2\alpha}\left(\left\|u(s)\right\|_{L^\infty(0,T)}^2 + \left\|f(\xi,u(\xi),u(q\xi))\right\|_{L^\infty(0,T)}^2\right) + cT\exp(T)h_n^{2(\alpha-\alpha_1)0}\left(\left\|u\right\|_{L^\infty(0,T)}^2 + \left\|(u(\xi)+u(q\xi))\right\|_{L^\infty(0,T)}^2\right) + cT\exp(T)h_1^{2m}M^{-2m}\left\|\partial_s^m u(s)\right\|_{L^1(I_1)}^2 + c\exp(T)\sum_{k=1}^{n-1}h_k^{2m}M_k^{-2m}\left(\left\|\partial_s^m f(s,u(s),u(qs))\right\|_{L^2(I_k)}^2 + \left\|\partial_s^m(u(s)+u(qs))\right\|_{L^2(I_k)}^2\right) \tag{35}$$

证明 由式 (18) 和引理 3～4 可得到

$$\max_{0\leqslant j\leqslant M}\left|e_{[0]}^n(t_{n,j})\right|^2 \leqslant c\sum_{k=1}^{n-1}h_k\max_{0\leqslant j\leqslant M}\left|e_{[0]}^n(t_{k,j})\right|^2 + ch_1^{2m}M^{-2m}\left\|\partial_s^m u(s)\right\|_{L^2(I_1)}^2 + ch_n^{2\alpha}\left(\left\|u(s)\right\|_{L^\infty(0,T)}^2 + \left\|f(\xi,u(\xi),u(q\xi))\right\|_{L^\infty(0,T)}^2\right) + ch_n^{2(\alpha-\alpha_1)}\left(\left\|u(s)\right\|_{L^\infty(0,T)}^2 + \left\|u(\xi)+u(q\xi)\right\|_{L^\infty(0,T)}^2\right) + ch_n^{2\alpha}\sum_{k=1}^{n-1}\max_{0\leqslant j\leqslant M}\left|e_{[0]}^k(t_{n,i})\right|^2 + ch_n^{2(\alpha-\alpha_1)}\sum_{k=1}^{n-1}\max_{0\leqslant j\leqslant M}\left|e_{[0]}^k(t_{n,i})\right|^2 + c\sum_{k=1}^{n-1}h_k^{2m}M^{-2m}\left(\left\|\partial_s^m f(s,u(s),u(qs))\right\|_{L^2(I_k)}^2 + \left\|\partial_s^m(u(s)+u(qs))\right\|_{L^2(I_k)}^2\right) \tag{36}$$

假设h_n足够的小，则

$$\max_{0\leqslant j\leqslant M}\left|e_{[0]}^n(t_{n,j})\right|^2 \leqslant c\sum_{k=1}^{n-1}h_k\max_{0\leqslant j\leqslant M}\left|e_{[0]}^k(t_{k,j})\right|^2 + ch_1^{2m}M^{-2m}\left\|\partial_s^m u\right\|_{L^2(I_1)}^2 + c\sum_{k=1}^{n-1}h_k^{2m}M^{-2m}\left(\left\|\partial_s^m f(t,u(s),u(qs))\right\|_{L^2(I_k)}^2 + \left\|\partial_s^m(u(s)+u(qs))\right\|_{L^2(I_k)}^2\right) + ch_n^{2\alpha}\left(\left\|u\right\|_{L^\infty(0,T)}^2 + \left\|f(\xi,u(\xi),u(q\xi))\right\|_{L^\infty(0,T)}^2\right) + ch_n^{2(\alpha-\alpha_1)}\left(\left\|u\right\|_{L^\infty(0,T)}^2 + \left\|u(\xi)+u(q\xi)\right\|_{L^\infty(0,T)}^2\right) \tag{37}$$

因为$h_k \leqslant ch_n$，由引理 2 可得

$$\max_{0\leqslant j\leqslant M}\left|e_{[0]}^n(t_{n,j})\right|^2 \leqslant cT\exp(T)h_n^{2\alpha}\left(\|u\|_{L^\infty(0,T)}^2+\|f(\xi,u(\xi),u(q\xi))\|_{L^\infty(0,T)}^2\right)+ cT\exp(T)h_n^{2(\alpha-\alpha_1)}\left(\|u(s)\|_{L^\infty(0,T)}^2+\|u(\xi)+u(q\xi)\|_{L^\infty(0,T)}^2\right)+ cT\exp(T)h_1^{2m}M^{-2m}\left\|\partial_s^m(u(s))\right\|_{L^2(I_1)}^2+c\exp(T)\sum_{k=1}^{n-1}h_k^{2m}M_k^{-2m}\left\|\partial_s^m f(t,u(s),u(qs))\right\|_{L^2(I_k)}^2 \tag{38}$$

3.2　矫正步骤的收敛性

本小节将对显示格式中的数值解式 (16) 在校正步骤中的收敛性进行分析。

引理 5　对于任意的整数$0\leqslant m\leqslant M$，都有

$$\max_{0\leqslant i\leqslant M}\left|e_{[l+1]}^n(t_{n,j})\right|^2 \leqslant cT\exp(T)h_1^{2m}M^{-2m}\left\|\partial_s^m u\right\|_{L^2(I_1)}^2+ch_n^{2m+2\alpha-1}M^{-2m}\left\|\partial_s^m f(t,u(s),u(qs))\right\|_{L^2(I_n)}^2+ ch_n^{2m+2(\alpha-\alpha_1)-1}M^{-2m}\left\|\partial_s^m(u(s)+u(qs))\right\|_{L^2(I_n)}^2+cT\exp(T)h_n^{2\alpha}\max_{\substack{0\leqslant k\leqslant n\\0\leqslant j\leqslant M}}\left|e_{[l]}^n(t_{k,j})\right|^2+ cT\exp(T)h_n^{2(\alpha-\alpha_1)}\max_{\substack{0\leqslant k\leqslant n\\0\leqslant j\leqslant M}}\left|e_{[l]}^n(t_{k,j})\right|^2+ c\exp(T)\sum_{k=1}^{n-1}h_k^{2m}M^{-2m}\left(\left\|\partial_s^m f(t,u(s),u(qs))\right\|_{L^2(I_k)}^2+\left\|\partial_s^m(u(s)+u(qs))\right\|_{L^2(I_k)}^2\right) \tag{39}$$

证明　由式 (13) 和式 (15) 可以得出

$$U_{[l+1]}^n(t_{n,j})=E_{[l]}^n(t_{n,j})+U_{[l]}^n(t_{n,j})=\frac{1}{\Gamma(\alpha)}\left(\sum_{k=1}^{n-1}\int_{I_k}(t_{n,j}-s)^{\alpha-1}\mathcal{I}_{s,M}^k f\left(s,U_{[l+1]}^k(s),U_{[l+1]}^{\hat{k}}(qs)\right)\right)\mathrm{d}s+ \int_{t_{n-1}}^{t_{n,j}}(t_{n,j}-\xi)^{\alpha-1}\mathcal{L}_\xi^n\left(f\left(\xi,U_{[l+1]}^n(\xi),\mathcal{L}_{q\xi}^n U_{[l+1]}^{\hat{n}}(q\xi)\right)-f\left(\xi,U_{[l]}^n(\xi),U_{[l]}^{\hat{n}}(q\xi)\right)\right)\mathrm{d}\xi+ \frac{1}{\Gamma(\alpha-\alpha_1)}\left(\sum_{k=1}^{n-1}\int_{I_k}(t_{n,j}-s)^{\alpha-\alpha_1-1}\left(U_{[l+1]}^k(s)+U_{[l+1]}^{\hat{k}}(qs)\right)\right)\mathrm{d}s+ \int_{t_{n-1}}^{t_{n,j}}(t_{n,j}-\xi)^{\alpha-\alpha_1-1}\left(\mathcal{L}_\xi^n\left(E_{[l]}^n(\xi)+E_{[l]}^n(q\xi)\right)+U_{[l]}^n(\xi)+U_{[l]}^n(q\xi)\right)\mathrm{d}\xi \tag{40}$$

结合式 (10) 可推出

$$e_{[l+1]}^n(t_{n,j})=u^n(t_{n,j})-U_{[l+1]}^n(t_{n,j})=\sum_{i=1}^5 D_i(t_{n,j}) \tag{41}$$

其中

$$D_1(t)=\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{n-1}\int_{I_k}(t-s)^{\alpha-1}\left(f\left(s,u^k(s),u^{\hat{k}}(qs)\right)-\mathcal{I}_{s,M}^k f\left(s,U_{[l+1]}^k(s),U_{[l+1]}^{\hat{k}}(qs)\right)\right)\mathrm{d}s$$

$$D_2(t)=\frac{1}{\Gamma(\alpha)}\int_{t_{n-1}}^t(t-\xi)^{\alpha-1}\left(f\left(\xi,u^n(\xi),u^{\hat{n}}(q\xi)\right)-\mathcal{I}_{s,M}^k f\left(\xi,U_{[l]}^n(\xi),U_{[l]}^{\hat{n}}(q\xi)\right)\right)\mathrm{d}\xi$$

$$D_3(t)=\frac{1}{\Gamma(\alpha)}\int_{t_{n-1}}^t(t-\xi)^{\alpha-1}\left(\mathcal{L}_\xi^n f\left(\xi,U_{[l]}^n(\xi),U_{[l]}^{\hat{n}}(q\xi)\right)-\mathcal{L}_\xi^n f\left(\xi,U_{[l+1]}^n(\xi),U_{[l+1]}^{\hat{n}}(q\xi)\right)\right)\mathrm{d}\xi$$

$$D_4(t)=\frac{1}{\Gamma(\alpha-\alpha_1)}\sum_{k=1}^{n-1}\int_{I_k}(t-s)^{\alpha-\alpha_1-1}\left(u^k(s)-U_{[l+1]}^k(s)+u^{\hat{k}}(qs)-U_{[l+1]}^{\hat{k}}(qs)\right)\mathrm{d}s$$

$$D_5(t)=\frac{1}{\Gamma(\alpha-\alpha_1)}\int_{t_{n-1}}^t(t-\xi)^{\alpha-\alpha_1-1}\left(u^n(\xi)+u^{\hat{n}}(q\xi)-U_{[l]}^n(\xi)-U_{[l]}^{\hat{n}}(q\xi)-\mathcal{L}_\xi^n E(\xi)-\mathcal{L}_\xi^{\hat{n}}E(q\xi)\right)\mathrm{d}\xi$$

显然，通过与引理 3 中类似的论证得到了以下估计：

$$\left|D_1(t_{n,j})\right|^2\leqslant c\sum_{k=1}^{n-1}h_k\max_{0\leqslant j\leqslant M}\left(\left|e_{[l+1]}^k(t_{k,j})\right|^2+ch_1^{2m}M^{-2m}\left\|\partial_s^m u(s)\right\|_{L^2(I_1)}^2\right)+c\sum_{k=1}^{n-1}h_k^{2m}M^{-2m}\left\|\partial_s^m f(t,u(s),u(qs))\right\|_{L^2(I_k)}^2 \tag{42}$$

分别估计$D_2(t_{n,j}), D_3(t_{n,j})$和$D_5(t_{n,j})$。由 Cauchy-Schwarz 不等式和三角形不等式，可以得到：

$$|D_2(t_{n,j})|^2 \leqslant \frac{1}{\Gamma^2(\alpha)}\int_{t_{n-1}}^{t_{n,j}}(t_{n,j}-\xi)^{2\alpha-2}\mathrm{d}\xi\int_{t_{n-1}}^{t_{n,j}}\left(f\left(\xi,u^n(\xi),u^{\hat{n}}(q\xi)\right)-\mathcal{I}_{\xi,M}^{k}f\left(\xi,U_{[l]}^{n}(\xi),U_{[l]}^{\hat{n}}(q\xi)\right)\right)^2\mathrm{d}\xi \leqslant$$
$$\frac{2}{\Gamma^2(\alpha)}\frac{h_n^{2\alpha-1}}{2\alpha-1}\left(\int_{t_{n-1}}^{t_{n,j}}\left(f\left(\xi,u^n(\xi),u^{\hat{n}}(q\xi)\right)-\mathcal{I}_{\xi,M}^{k}f\left(\xi,u^n(\xi),u^{\hat{n}}(q\xi)\right)\right)^2\mathrm{d}\xi+\right.$$
$$\left.\int_{I_n}\mathcal{I}_{\xi,M}^{n}\left(f\left(\xi,u^n(\xi),u^{\hat{n}}(q\xi)\right)-f\left(\xi,U_{[l]}^{n},U_{[l]}^{\hat{n}}(q\xi)\right)\right)\right)^2\mathrm{d}\xi \tag{43}$$

将引理 1、式 (5) 和式 (17) 应用到式 (43) 中可以推出

$$|D_2(t_{n,j})|^2 \leqslant ch_n^{2m+2\alpha-1}M_n^{-2m}\left\|\partial_s^m f(t,u(s),u(qs))\right\|_{L^\infty(0,T)}^2+ch_n^{2\alpha}\max_{0\leqslant i\leqslant M}\left(\left|e_{[l]}^{n}(t_{n,i})\right|^2+\left|e_{[l]}^{\hat{n}}(qt_{n,i}))\right|^2\right)+$$
$$ch_n^{2m+2\alpha-1}M^{-2m}\left\|\partial_t^m u\right\|_{L^2(I_1)}^2 \tag{44}$$

与式 (34) 类似，得

$$\left|D_3(t_{n,j})\right|^2 \leqslant ch_n^{2\alpha}\left(\max_{0\leqslant i\leqslant M}\left(\left|e_{[l]}^{n}(t_{n,i})\right|^2+\max_{0\leqslant i\leqslant M}\left(\left|e_{[l+1]}^{n}(t_{n,i})\right|^2\right)+ch_n^{2\alpha-1}\max_{\substack{0\leqslant i\leqslant M\\0\leqslant k\leqslant n-1}}h_k\left(\left|e_{[l]}^{n}(t_{n,i})\right|^2+\left|e_{[l+1]}^{n}(t_{n,i})\right|^2\right)\right)\right) \tag{45}$$

与式 (23)、式 (31) 类似，可以推导出

$$|D_4(t_{n,j})|^2 \leqslant c\sum_{k=1}^{n-1}h_k\max_{0\leqslant j\leqslant M}\left(\left|e_{[l]}^{k}(t_{k,j})\right|^2\right)+c\sum_{k=1}^{n-1}h_k^{2m}M^{-2m}\left\|\partial_s^m(u(s)+u(qs))\right\|_{L^2(I_k)}^2+ch_1^{2m}M^{-2m}\left\|\partial_s^m u(s)\right\|_{L^2(I_1)}^2 \tag{46}$$

$$\left|D_5(t_{n,j})\right|^2 \leqslant ch_n^{2(\alpha-\alpha_1)}\left(\max_{0\leqslant i\leqslant M}\left(\left|e_{[l]}^{n}(t_{n,i})\right|^2+\max_{0\leqslant i\leqslant M}\left|e_{[l+1]}^{\widehat{n}}(t_{n,i})\right|^2\right)+\right.$$
$$ch_n^{2(\alpha-\alpha_1)}\left(h_k\max_{0\leqslant i\leqslant M}\left(\left|e_{[l]}^{n}(t_{k,i})\right|^2+h_k\max_{0\leqslant i\leqslant M}\left|e_{[l+1]}^{\widehat{n}}(t_{k,i})\right|^2\right)\right)+$$
$$ch_n^{2m+2(\alpha-\alpha_1)-1}\left(\left\|\partial_s^m(u(s)+u(qs))\right\|_{L^\infty(0,T)}^2\right)+ch_n^{2m+2(\alpha-\alpha_1)-1}M^{-2m}\left\|\partial_t^m u\right\|_{L^2(I_1)}^2 \tag{47}$$

结合式 (42)～(47) 可以得到

$$\max_{0\leqslant i\leqslant M}\left|e_{[l+1]}^{n}(t_{n,j})\right|^2 \leqslant c\sum_{k=1}^{n-1}h_k\max_{0\leqslant j\leqslant M}\left|e_{[l+1]}^{k}(t_{k,j})\right|^2+c\sum_{k=1}^{n-1}h_k^{2m}M^{-2m}\left\|\partial_s^m f(s,u(s),u(qs))\right\|_{L^2(I_k)}^2+$$
$$c\sum_{k=1}^{n-1}h_k^{2m}M^{-2m}\left\|\partial_s^m u\right\|_{L^2(I_k)}^2+ch_n^{2m+2\alpha-1}M^{-2m}\left\|\partial_s^m f(s,u(s),u(qs))\right\|_{L^\infty(0,T)}^2+$$
$$c\sum_{k=1}^{n-1}h_k^{2m}M^{-2m}\left\|\partial_s^m(u(s)+u(qs))\right\|_{L^2(I_k)}^2+ch_n^{2m+2(\alpha-\alpha_1)-1}M^{-2m}\left\|\partial_s^m(u(s)+u(qs))\right\|_{L^\infty(0,T)}^2+$$
$$c\left(h_n^{2\alpha}+h_n^{2(\alpha-\alpha_1)}\right)\max_{0\leqslant i\leqslant M_n}\left|e_{[l+1]}^{k}(t_{k,i})\right|^2+c\left(h_n^{2\alpha}+h_n^{2(\alpha-\alpha_1)}\right)h_k\max_{\substack{0\leqslant j\leqslant M\\1\leqslant k\leqslant n}}\left|e_{[l]}^{n}(t_{k,j})\right|^2 \tag{48}$$

假设h_n足够小，由式 (48) 可以得到

$$\max_{0\leqslant i\leqslant M}\left|e_{[l+1]}^{n}(t_{n,j})\right|^2 \leqslant c\sum_{k=1}^{n-1}h_k\max_{0\leqslant j\leqslant M}\left|e_{[l+1]}^{k}(t_{k,j})\right|^2+c\sum_{k=1}^{n-1}h_k^{2m}M^{-2m}\left\|\partial_s^m f(t,u(s),u(qs))\right\|_{L^2(I_k)}^2+$$
$$c\sum_{k=1}^{n-1}h_k^{2m}M^{-2m}\left\|\partial_s^m u(s)\right\|_{L^2(I_1)}^2+ch_n^{2m+2\alpha-1}M_n^{-2m}\left\|\partial_s^m f(t,u(s),u(qs))\right\|_{L^2(I_k)}^2+$$
$$ch_n^{2m+2(\alpha-\alpha_1)-1}M_n^{-2m}\left\|\partial_s^m(u(s)+u(qs))\right\|_{L^2(I_k)}^2+ch_n^{2m+2(\alpha-\alpha_1)-1}M_n^{-2m}\left\|\partial_s^m(u(s)+u(qs))\right\|_{L^\infty(0,T)}^2+$$
$$ch_n^{2(\alpha-\alpha_1)-1}\left\|u(t)\right\|_{L^\infty(0,T)}^2+\left(ch_n^{2\alpha-1}+ch_n^{2(\alpha-\alpha_1)-1}\right)\max_{\substack{0\leqslant j\leqslant M\\0\leqslant k\leqslant n}}\left|e_{[l]}^{n}(t_{n,j})\right|^2 \tag{49}$$

由于$h_k \leqslant ch_n\,(k\leqslant n)$，结合引理 2 得到

$$\max_{0\leqslant i\leqslant M}\left|e^n_{[l+1]}(t_{n,j})\right|^2\leqslant ch_n^{2m+2\alpha-1}M^{-2m}\left\|\partial_s^m f(t,u(s),u(qs))\right\|^2_{L^\infty(0,T)}+ch_n^{2m+2(\alpha-\alpha_1)-1}M^{-2m}\left\|\partial_s^m(u(s)+u(qs))\right\|^2_{L^\infty(0,T)}+$$
$$c\exp(T)\sum_{k=1}^{n-1}h_k^{2m}M^{-2m}\left(\left\|\partial_s^m f(t,u(s),u(qs))\right\|^2_{L^2(I_k)}+\left\|\partial_s^m(u(s)+u(qs))\right\|^2_{L^2(I_k)}\right)+$$
$$c\exp(T)\sum_{k=1}^{n-1}h_k^{2m}M^{-2m}\left\|\partial_t^m u(t)\right\|^2_{L^2(I_1)}+cT\exp(T)\left(h_n^{2\alpha}+h_n^{2(\alpha-\alpha_1)}\right)\max_{0\leqslant i\leqslant M_n}\left|e^n_{[l]}(t_{n,i})\right|^2 \tag{50}$$

由此完成了证明。

4　数值实验

通过数值实验来说明所提出算法的效率。将网格点处的离散L^∞误差定义为

$$E_{L^\infty(0,T)}:=\max_{\substack{1\leqslant k\leqslant N\\0\leqslant j\leqslant M}}|u(t_{k,j})-U(t_{k,j})| \tag{51}$$

例 1　考虑以下分数阶非线性微分方程：

$$\begin{cases}D^{\frac{3}{2}}u(t)=u(t)+u\left(\dfrac{t}{2}\right)+D^{\frac{1}{4}}u(t)+D^{\frac{1}{4}}u\left(\dfrac{t}{2}\right)\\u(0)=0,u'(0)=0\end{cases} \tag{52}$$

其真解为

$$u(t)=t^{2.5} \tag{53}$$

测试问题 (52) 的显示格式式 (11)、式 (16) 的数值误差。采取多种$N^{\rm c}$(粗划分网格的数量) 和$N^{\rm f}$(细化分网格的数量)，设置均匀的矫正步长为L， SDC 方法中$M=2$, $M=4$, $M=6$, $M=8$对应的数值误差$E^\infty(0,T)$在图 1(a)～(d) 中显示。可以观察到，数值误差随着h的减少而衰减，并且收敛速度随着M和L的增加而变快。

数值结果表明，随着校正过程的增加，SDC 格式的收敛率都显著增加。另外，数值误差随着h的减小而衰减。这意味着可以通过细化网格或增加多项式的次数来减小误差，但由于最高收敛速度$h_n^mM_n^{-m}$的限制，增长速度随着校正过程数量的增加逐渐放缓，这与定理 2 中的理论分析非常吻合。

例 2 考虑以下分数阶非线性偏微分方程：

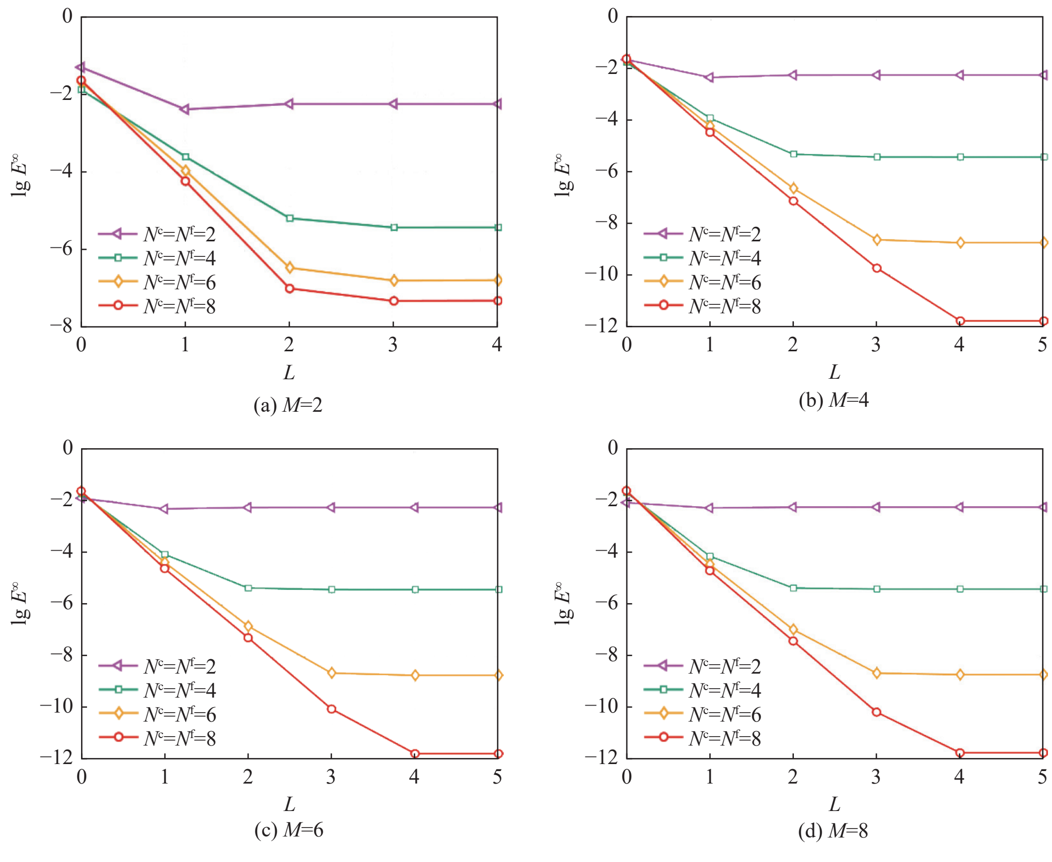


图 1　问题 (52) 在L^∞范数下的数值误差

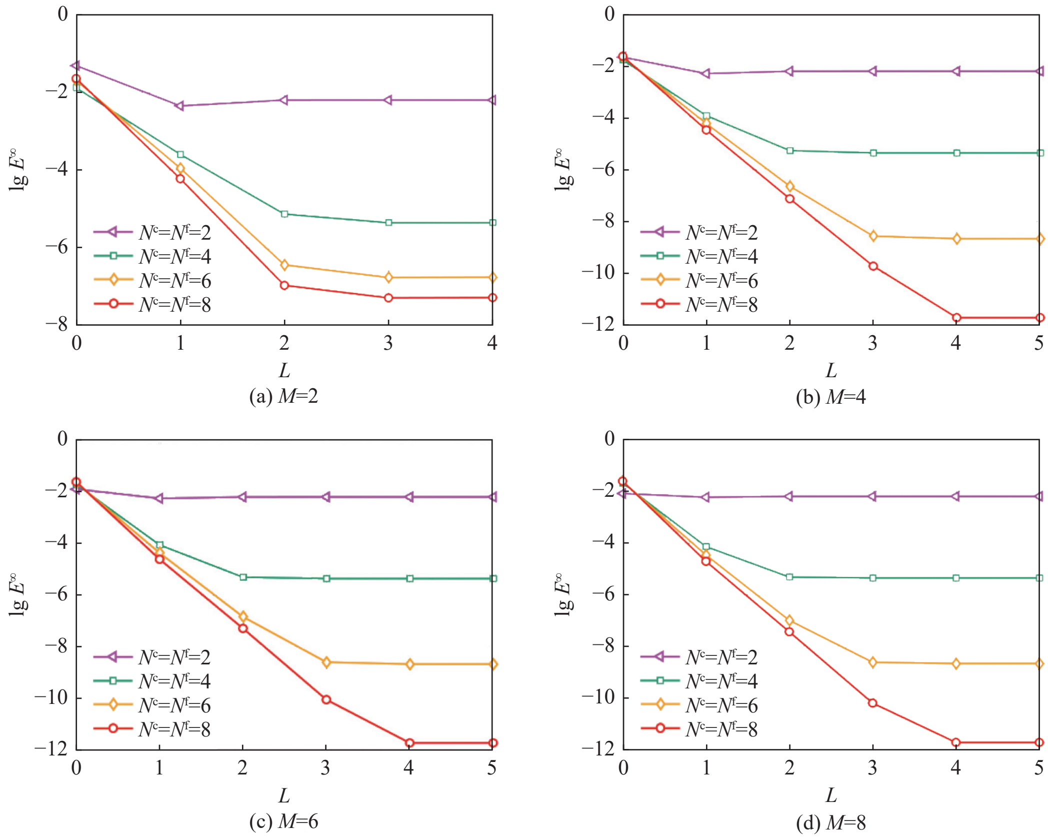
Fig.1　Numerical errors under L^∞ for question (52)

$$\begin{cases} D^{\frac{3}{2}}u(t) = u(t) + u\left(\frac{t}{2}\right) + \\ \quad D^{\frac{1}{4}}u(t) + D^{\frac{1}{4}}u\left(\frac{t}{2}\right) - t^2 - 2t - 3 \\ u(0) = 0, u'(0) = 0 \end{cases} \tag{54}$$

其真解为

$$u(t) = t^{2.5} \tag{55}$$

测试问题 (54) 的显示格式式 (11)、式 (16) 的数值误差。采取多种 N^c 和 N^f，设置均匀的矫正步长 L。SDC 方法中 $M = 2, 4, 6, 8$ 相应的数值误差 $E^\infty(0,T)$ 在图 2(a)～(d) 中显示。再次观察到，数值误差随着 h 的减少而衰减，并且随着 M 和 L 的增加而变快。



(a) M=2 (b) M=4 (c) M=6 (d) M=8

图 2 问题 (54) 在 L^∞ 范数下的数值误差

Fig.2 Numerical errors under L^∞ for question (54)

5 结 论

本文成功提出了一种崭新的网格划分方式，涵盖全局网格与局部网格的巧妙设计。在全局网格的构建上，确保每个子区间在经历延迟后，能够精确地与前一个子区间对齐，从而有效降低由于延迟引发的子区间之间的相互影响。同时，局部网格采用勒让德高斯点，极大地提升了谱延迟矫正法（SDC）的稳定性。此外，本文引入显式双网格 SDC 算法，该算法在保持出色稳定性的同时，允许灵活调整全局和局部网格的划分方式，以满足不同的精度需求。最后，本文对 SDC 算法的预测与校正过程进行了严格的收敛性分析，并通过数值实验有力地验证了这一分析结果的准确性。这些突出贡献为提升 SDC 方法的稳定性和灵活性提供了切实有效的解决方案。

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